Cons-free Rewriting

Non-determinism in implicit computational complexity

Cynthia Kop; joint work with Jakob Grue Simonsen

26 June, 2019

IMPLICIT COMPLEXITY

Complexity Classes

Complexity Classes

- classes like P

Complexity Classes

- classes like P, NP

Complexity Classes

- classes like P, NP, EXP

Complexity Classes

- classes like P, NP, EXP, LOGSPACE

Complexity Classes

- classes like P, NP, EXP, LOGSPACE, ...

Complexity Classes

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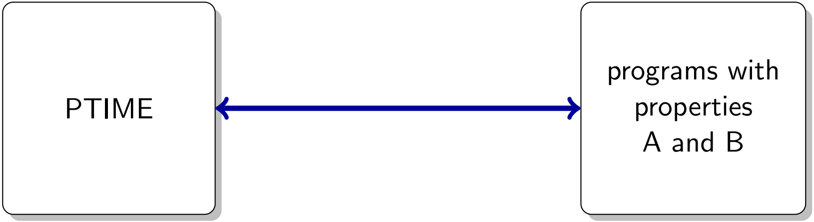
decision problems which can be decided in polynomial time

Characterising Classes

Characterising Classes



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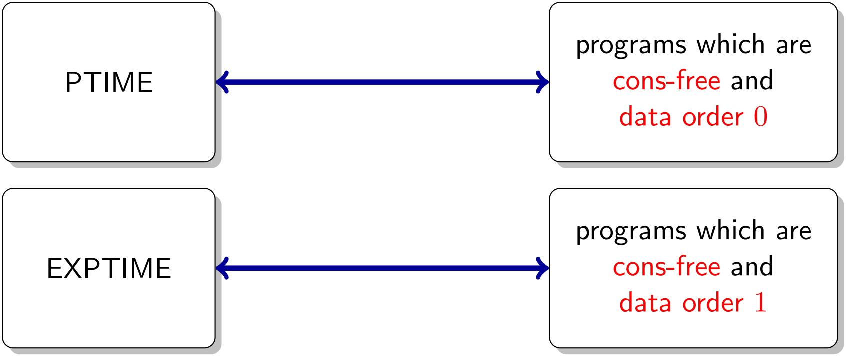
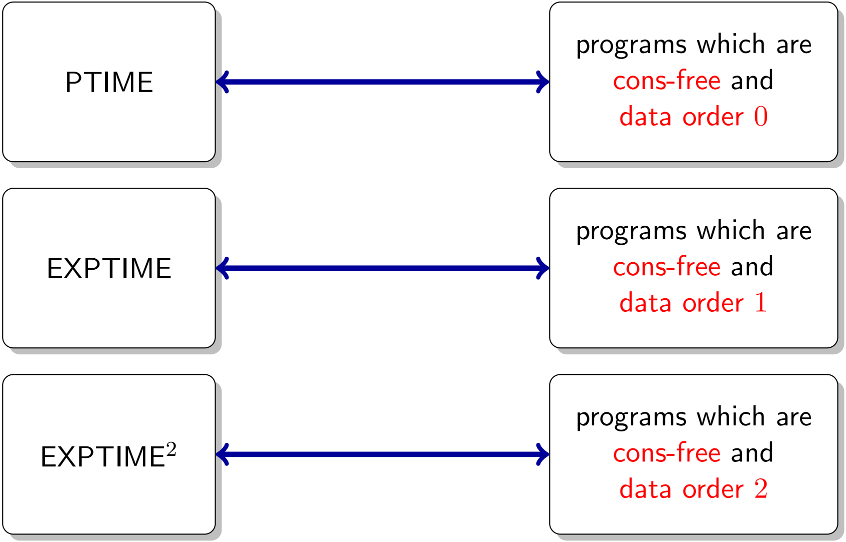
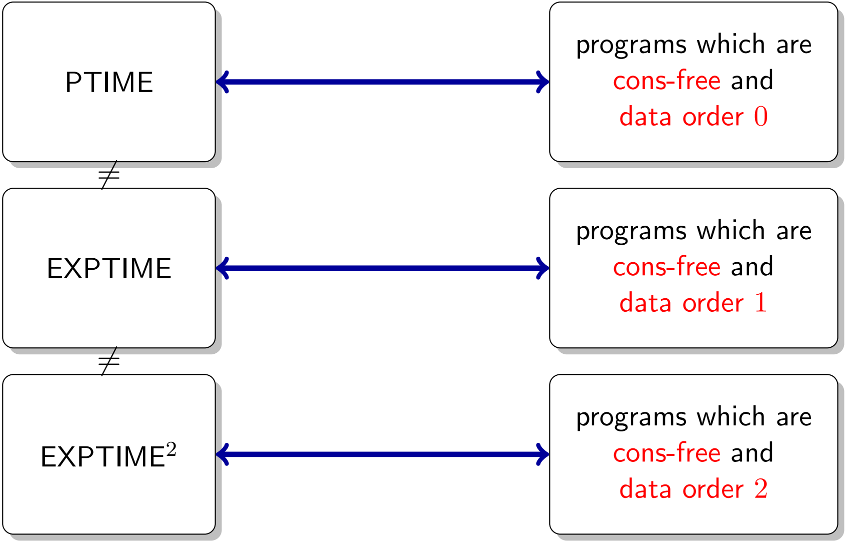
PTIME ⟷ programs which are cons-free and data order 0


PTIME
programs which are cons-free and data order 0
EXPTIME
programs which are cons-free and data order 1


PTIME
programs which are cons-free and data order 0
EXPTIME
programs which are cons-free and data order 1
EXPTIME²
programs which are cons-free and data order 2


PTIME
programs which are cons-free and data order 0
≠
EXPTIME
programs which are cons-free and data order 1
≠
EXPTIME^2
programs which are cons-free and data order 2

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Why not term rewriting?

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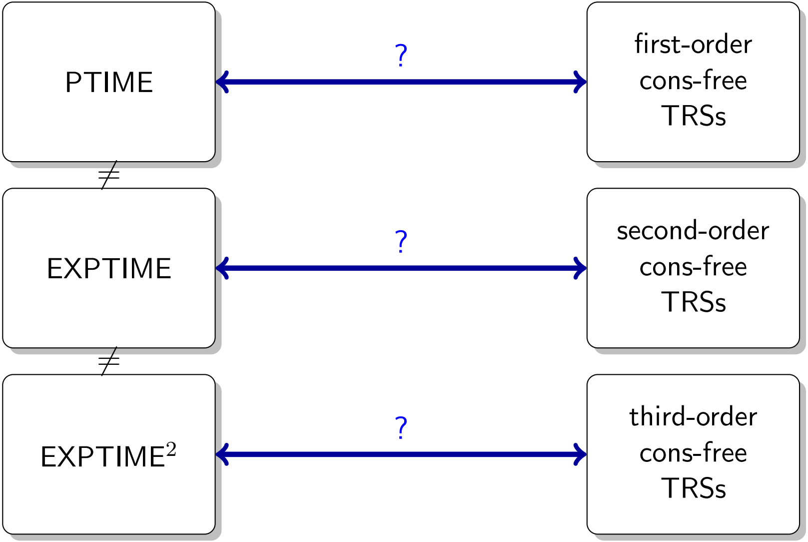
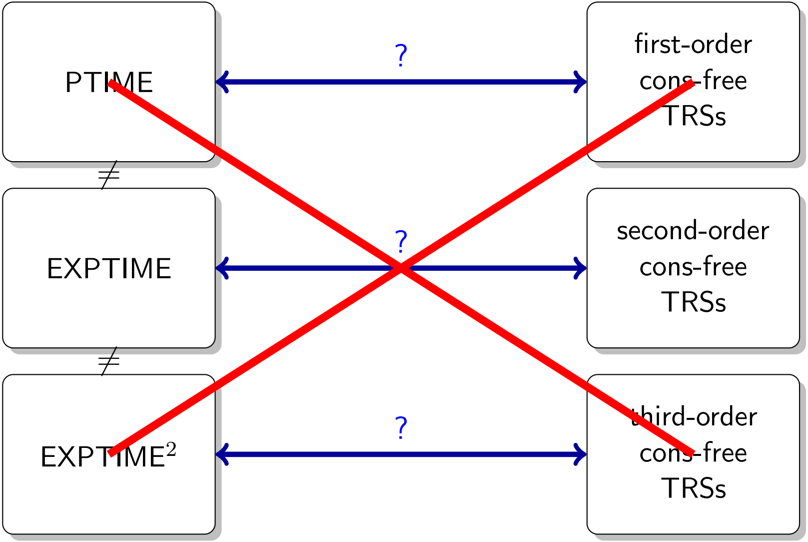
Let's see what comes out!

PTIME	⟷	programs which are cons-free and data order 0
$\neq$		
EXPTIME	⟷	programs which are cons-free and data order 1
$\neq$		
EXPTIME^2	⟷	programs which are cons-free and data order 2


PTIME
?
first-order cons-free TRSs
≠
EXPTIME
?
second-order cons-free TRSs
≠
EXPTIME^2
?
third-order cons-free TRSs


PTIME
≠
EXPTIME
≠
EXPTIME^2
?
?
?
first-order cons-free TRSs
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In 2001, Jones

- showed that cons-free programs with data order K characterise $\mathsf{EXPTIME}^K$

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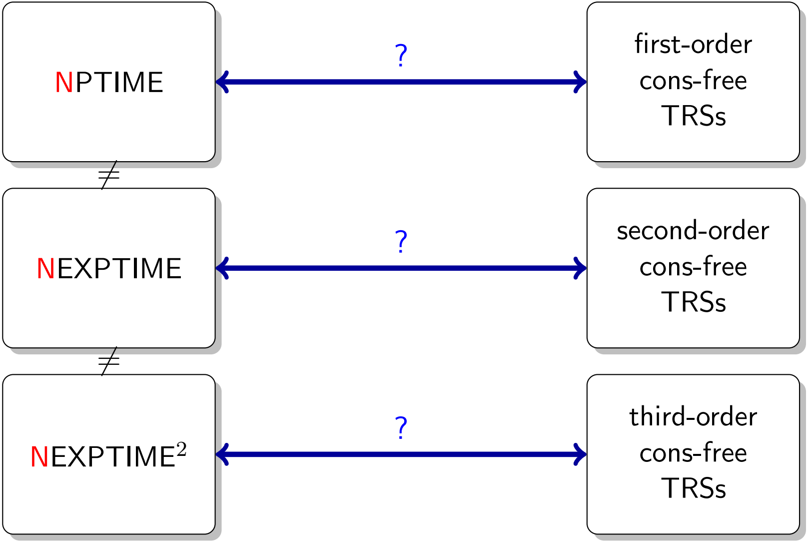
- showed that deterministic cons-free programs with data order K and call-by-value reduction characterise $\mathsf{EXPTIME}^K$


NPTIME
?
first-order cons-free TRSs
≠
NEXPTIME
?
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≠
NEXPTIME^2
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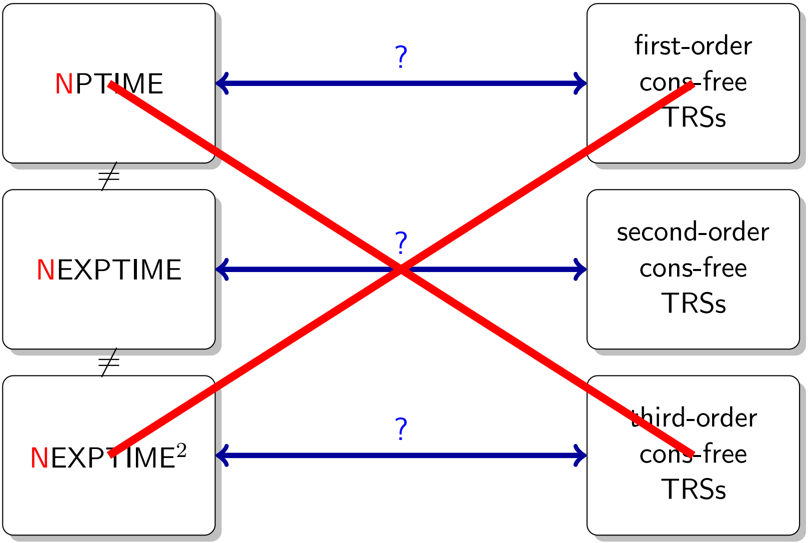
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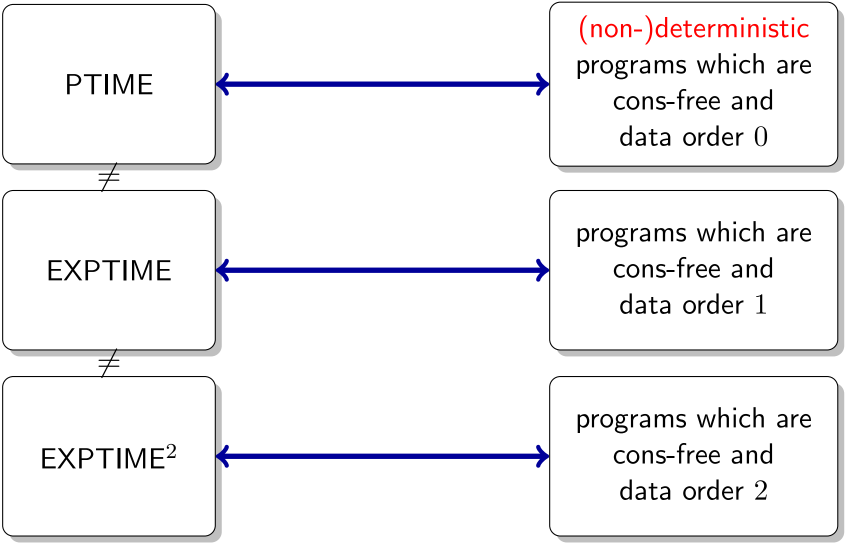
- showed that non-deterministic cons-free programs with data order 0 and call-by-value reduction ...

In 2001, Jones

- showed that deterministic cons-free programs with data order K and call-by-value reduction characterise EXPTIME^K

In 2006, Bonfante

- showed that non-deterministic cons-free programs with data order 0 and call-by-value reduction characterise P $=$ EXPTIME^0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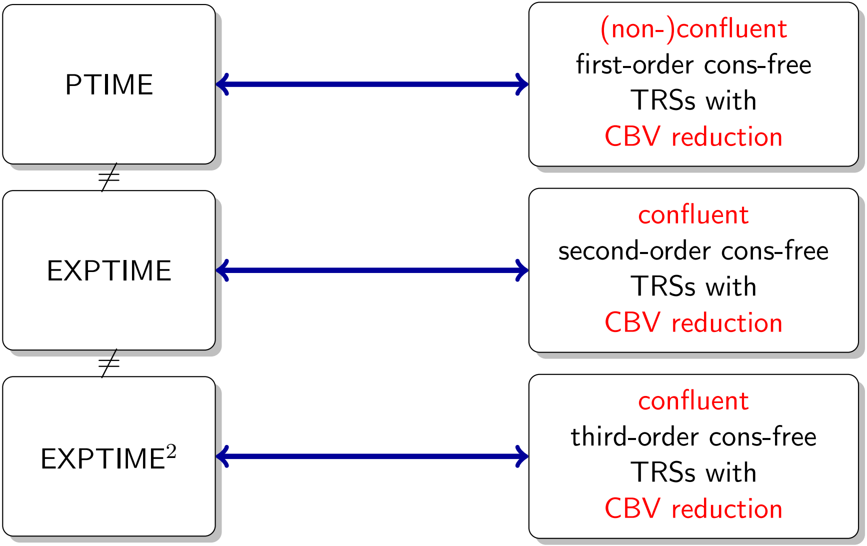
PTIME
(non-)confluent
first-order cons-free
TRSs with
CBV reduction
≠
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EXPTIME^2
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Overview

1. cons-free applicative rewriting
   (what is this "cons-freeness" and how do we use it?)
2. characterisations with first-order cons-free innermost rewriting
   (the general idea)
3. characterisations with higher-order cons-free innermost rewriting
   (where it starts to get interesting)
4. characterisations using non-innermost cons-free rewriting
   (where it really gets interesting)

CONS-FREE TERM REWRITING

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Basics

- applicative rewriting with simple types

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- all rules must be constructor rules
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- in Jones: call-by-value reduction; values are:
  - ground expressions built from constructors (data)
  - incomplete function applications f $v_1 \cdots v_n$ with all v_i values

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$$
\begin{array}{rcl}
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\mathsf{Helper}\ []\ b & = & b \\
\mathsf{Helper}\ (h{::}t)\ \mathsf{True} & = & \mathsf{Helper}\ t\ \mathsf{False} \\
\mathsf{Helper}\ (h{::}t)\ \mathsf{False} & = & \mathsf{Helper}\ t\ \mathsf{True}
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Cons-free counting

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Traditional:

$$\begin{array}{rcl} \text{Succ } x & = & \text{S } x \\ \text{Pred } 0 & = & 0 \\ \text{Pred } (\text{S } x) & = & x \\ \text{Add } 0\ y & = & 0 \\ \text{Add } (\text{S } x)\ y & = & \text{Succ } (\text{Add } x\ y) \\ & \dots & \end{array}$$

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\dots & &
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$$\begin{aligned} \text{Succ } [n] &= [\min(n+1, \text{length}(input))] \\ \text{Pred } [n] &= [\max(n-1, 0)] \\ \text{Equals } [n]\ [m] &= [\text{True if } n = m \text{ and False otherwise}] \end{aligned}$$

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$$

Idea:

$$
\begin{array}{rcl}
\mathsf{Succ}\ inp\ x & = & \\
\mathsf{Pred}\ inp\ [] & = & 0 \\
\mathsf{Pred}\ inp\ (x{::}xs) & = & xs \\
\mathsf{Add}\ inp\ []\ y & = & [] \\
\mathsf{Add}\ inp\ (x{::}xs)\ y & = & \mathsf{Succ}\ inp\ (\mathsf{Add}\ inp\ xs\ y) \\
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Idea:

$$\begin{array}{rcl} \mathsf{Iterate}\ inp\ z\ x & = & \mathsf{Test}\ inp\ z\ x\ (\mathsf{Equals}\ (\mathsf{Pred}\ z)\ x) \\ \mathsf{Test}\ inp\ z\ x\ \mathsf{True} & = & z \\ \mathsf{Test}\ inp\ z\ x\ \mathsf{False} & = & \mathsf{Iterate}\ inp\ (\mathsf{Pred}\ z) \\ \mathsf{Succ}\ inp\ x & = & \mathsf{Iterate}\ inp\ inp\ x \\ \mathsf{Pred}\ inp\ [] & = & \mathsf{0} \\ \mathsf{Pred}\ inp\ (x{::}xs) & = & xs \\ \mathsf{Add}\ inp\ []\ y & = & [] \\ \mathsf{Add}\ inp\ (x{::}xs)\ y & = & \mathsf{Succ}\ inp\ (\mathsf{Add}\ inp\ xs\ y) \\ \dots & & \end{array}$$

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Wish:

$$\begin{array}{rcl} \mathsf{Succ}\ [n] & = & [\min(n+1, 5 \cdot (\mathsf{length}(\mathit{list})+1)^2-1)] \\ \mathsf{Pred}\ [n] & = & [\max(n-1, 0)] \\ \mathsf{Equals}\ [n]\ [m] & = & [\mathsf{True} \text{ if } n = m \text{ and } \mathsf{False} \text{ otherwise}] \end{array}$$

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Idea:

$$\text{Seed } \mathit{list} \quad = \quad (\mathit{list},\ 0{::}0{::}0{::}0{::}[\,], \mathit{list}, \mathit{list})$$

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\end{aligned}
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Idea:

$$\text{Seed } \mathit{list} = (\mathit{list},\ 0{::}0{::}0{::}0{::}[], \mathit{list}, \mathit{list})$$

Corresponds to: $|0{::}0{::}0{::}0{::}[]| * (n+1)^2 + |\mathit{list}| * (n+1)^1 + |\mathit{list}|$

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Idea:

$$\begin{aligned}
\text{Seed}^1\ \mathit{list} &= 0{::}0{::}0{::}0{::}[] \\
\text{Seed}^2\ \mathit{list} &= \mathit{list} \\
\text{Seed}^3\ \mathit{list} &= \mathit{list} \\
\text{Pred}^1\ \mathit{list}\ xs\ ys\ (z{::}zs) &= xs \\
\text{Pred}^2\ \mathit{list}\ xs\ ys\ (z{::}zs) &= ys \\
\text{Pred}^3\ \mathit{list}\ xs\ ys\ (z{::}zs) &= zs \\
&\dots
\end{aligned}$$

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Idea:

- a value $F : \mathsf{list}^4 \Rightarrow \mathsf{bool}$ describes a bitstring
- use bitvector arithmetic to calculate successor and predecessor

Cons-free counting

$\Longrightarrow$ using variables with type order K we can count up to $\exp_2^K(a \cdot n^b)$

CHARACTERISING COMPLEXITY CLASSES

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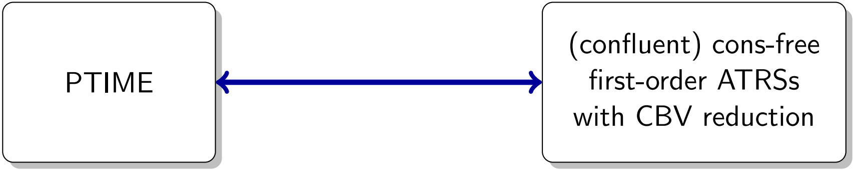
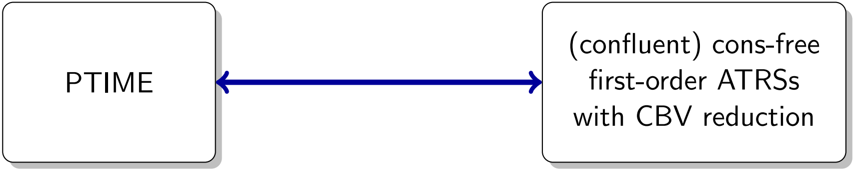
2. characterisations with first-order cons-free innermost rewriting
   (the general idea)
3. characterisations with higher-order cons-free innermost rewriting
   (where it starts to get interesting)
4. characterisations using non-innermost cons-free rewriting
   (where it really gets interesting)

How to prove a characterisation?

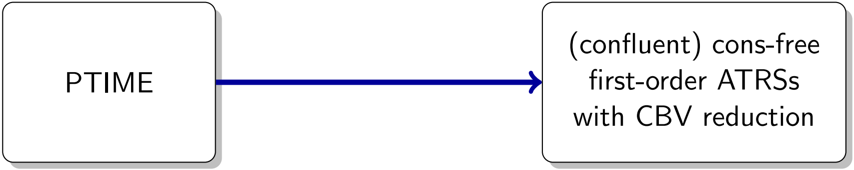


How to prove a characterisation?



For every decision problem X:

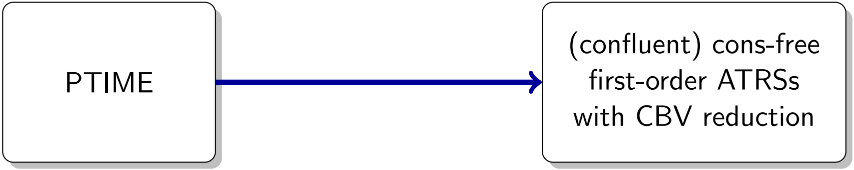
How to prove a characterisation?



For every decision problem X:

- if $X \in$ PTIME then there is a confluent cons-free first-order ATRS which accepts X with CBV evaluation

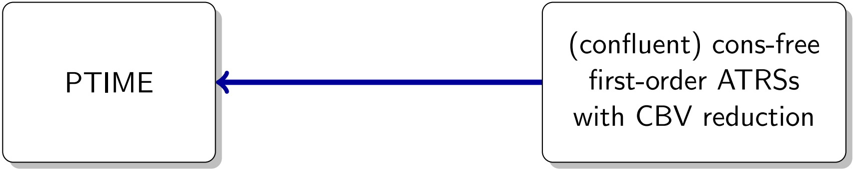
How to prove a characterisation?



For every decision problem X:

- the final state of any given Turing Machine operating in polynomial time can be found by a confluent cons-free first-order ATRS using CBV reduction

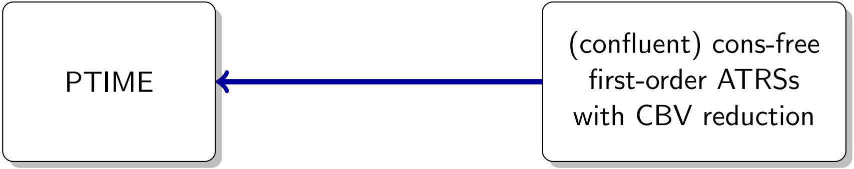
How to prove a characterisation?



For every decision problem X:

- the final state of any given Turing Machine operating in polynomial time can be found by a confluent cons-free first-order ATRS using CBV reduction
- if there is a (confluent or non-confluent) cons-free first-order ATRS which accepts X using call-by-value reduction, then $X \in$ PTIME

How to prove a characterisation?



For every decision problem X:

- the final state of any given Turing Machine operating in polynomial time can be found by a confluent cons-free first-order ATRS using CBV reduction
- the result of any given cons-free first-order ATRS with CBV evaluation can be found by an algorithm operating in polynomial time

A problem?

Perfectly allowed: $\mathsf{F}\ (\mathsf{S}\ x) \to \mathsf{G}\ (\mathsf{F}\ x)\ (\mathsf{F}\ x)$

A problem?

Perfectly allowed: $\mathsf{F}\ (\mathsf{S}\ x) \rightarrow \mathsf{G}\ (\mathsf{F}\ x)\ (\mathsf{F}\ x)$

$\Longrightarrow$ use caching!

Constructing an algorithm

Constructing an algorithm

Rules:

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

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Start term: $\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[])$

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Let $\mathcal{B} =$

Constructing an algorithm

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\end{array}
\qquad
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Make a list:

Constructing an algorithm

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Make a list:

$$\mathsf{Subseq}\ (0;0;1;[])\ (0;0;1;[]) \quad \to^* \quad \{\}$$

Constructing an algorithm

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Make a list:

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Statements:

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\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
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\end{array}
$$

Statements:

$$
\begin{array}{rl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* \\
\dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* \\
\mathsf{Subseq}\ []\ [] & \to^* \\
\dots &
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
& \dots &
\end{array}
$$

$$\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
 & \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \ldots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \ldots & \end{array}$$

$$\begin{array}{rcl@{\quad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x::t) & \to & t & \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
& \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rclcrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
 & \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
& \dots &
\end{array}
$$

$$\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x::t) & \to & t & \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
 & \dots &
\end{array}$$

$$\begin{array}{rcl} \text{Subseq}\ []\ t & \to & \text{True} \\ \text{Subseq}\ s\ [] & \to & \text{False} \\ \text{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \text{Subseq}\ s\ t & \to & \text{Subseq}\ s\ (\text{Tl}\ t) \\ \text{Subseq}\ (0{::}s)\ (0{::}t) & \to & \text{Subseq}\ s\ t \\ \text{Subseq}\ (1{::}s)\ (1{::}t) & \to & \text{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \text{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \text{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \text{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \text{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \text{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \text{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
& \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$
\begin{array}{rcl@{\quad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x::t) & \to & t & \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
& \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
 & \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \text{Subseq}\ []\ t & \to & \text{True} \\ \text{Subseq}\ s\ [] & \to & \text{False} \\ \text{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \text{Subseq}\ s\ t & \to & \text{Subseq}\ s\ (\text{Tl}\ t) \\ \text{Subseq}\ (0{::}s)\ (0{::}t) & \to & \text{Subseq}\ s\ t \\ \text{Subseq}\ (1{::}s)\ (1{::}t) & \to & \text{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \text{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \text{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \text{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \text{False} \\ \text{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \text{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \text{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
 & \dots &
\end{array}$$

$$\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \\
& \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \end{array}$$

. . .

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \text{Subseq}\ []\ t & \to & \text{True} \\ \text{Subseq}\ s\ [] & \to & \text{False} \\ \text{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \text{Subseq}\ s\ t & \to & \text{Subseq}\ s\ (\text{Tl}\ t) \\ \text{Subseq}\ (0{::}s)\ (0{::}t) & \to & \text{Subseq}\ s\ t \\ \text{Subseq}\ (1{::}s)\ (1{::}t) & \to & \text{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \text{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \text{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \text{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \text{False} \\ \text{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \text{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \text{Subseq}\ (1;[])\ (1;[]) & \to^* & \\ \text{Subseq}\ []\ [] & \to^* & \text{True, False} \\ & \dots & \end{array}$$

$$\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}$$

$$\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x::t) & \to & t & \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \end{array}$$

. . .

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl}
\text{Subseq}\ []\ t & \to & \text{True} \\
\text{Subseq}\ s\ [] & \to & \text{False} \\
\text{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\text{Subseq}\ s\ t & \to & \text{Subseq}\ s\ (\text{Tl}\ t) \\
\text{Subseq}\ (0{::}s)\ (0{::}t) & \to & \text{Subseq}\ s\ t \\
\text{Subseq}\ (1{::}s)\ (1{::}t) & \to & \text{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\text{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\text{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\text{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\text{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\text{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \text{False} \\
\text{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\text{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\text{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \text{True, False} \\
\text{Subseq}\ (1;[])\ (1;[]) & \to^* & \text{True, False} \\
\text{Subseq}\ []\ [] & \to^* & \text{True, False} \\
 & \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \end{array}$$

...

$$\begin{array}{rcl}
\text{Subseq}\ []\ t & \to & \text{True} \\
\text{Subseq}\ s\ [] & \to & \text{False} \\
\text{Tl}\ (x::t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\text{Subseq}\ s\ t & \to & \text{Subseq}\ s\ (\text{Tl}\ t) \\
\text{Subseq}\ (0::s)\ (0::t) & \to & \text{Subseq}\ s\ t \\
\text{Subseq}\ (1::s)\ (1::t) & \to & \text{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\text{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\text{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\text{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\text{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\text{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \text{False} \\
\text{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\text{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\text{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \text{True}, \text{False} \\
\text{Subseq}\ (1;[])\ (1;[]) & \to^* & \text{True}, \text{False} \\
\text{Subseq}\ []\ [] & \to^* & \text{True}, \text{False} \\
& \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x::t) & \to & t & \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x::t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t \\
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}
$$

$$\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False}
\end{array}
$$

. . .

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x::t) & \to & t & \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t \\
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rclrcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}
$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\
\mathsf{Tl}\ (x::t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False}
\end{array}
$$

...

$$\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}$$

Statements:

$$\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
& \dots & \\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
& \dots &
\end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{aligned}
\mathsf{Subseq}\ []\ t &\to \mathsf{True} & \mathsf{Subseq}\ s\ t &\to \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t)\\
\mathsf{Subseq}\ s\ [] &\to \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) &\to \mathsf{Subseq}\ s\ t\\
\mathsf{Tl}\ (x{::}t) &\to t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) &\to \mathsf{Subseq}\ s\ t
\end{aligned}$$

Statements:

$$\begin{aligned}
\mathsf{Tl}\ (0;1;0;1;[]) &\to^* 1;0;1;[]\\
\mathsf{Tl}\ (1;0;1;[]) &\to^* 0;1;[]\\
&\dots\\
\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) &\to^* \mathsf{True}, \mathsf{False}\\
\mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) &\to^* \mathsf{False}\\
\mathsf{Subseq}\ (0;0;1;[])\ ([]) &\to^* \mathsf{False}\\
\mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) &\to^* \mathsf{True}, \mathsf{False}\\
\mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) &\to^* \mathsf{True}, \mathsf{False}\\
\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) &\to^* \mathsf{True}, \mathsf{False}\\
\mathsf{Subseq}\ (1;[])\ (1;[]) &\to^* \mathsf{True}, \mathsf{False}\\
\mathsf{Subseq}\ []\ [] &\to^* \mathsf{True}, \mathsf{False}\\
&\dots
\end{aligned}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x{::}t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$\begin{array}{rcl} \mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\ \mathsf{Subseq}\ s\ [] & \to & \mathsf{False} \\ \mathsf{Tl}\ (x::t) & \to & t \end{array} \qquad \begin{array}{rcl} \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\ \mathsf{Subseq}\ (0::s)\ (0::t) & \to & \mathsf{Subseq}\ s\ t \\ \mathsf{Subseq}\ (1::s)\ (1::t) & \to & \mathsf{Subseq}\ s\ t \end{array}$$

Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ s\ [] & \to & \mathsf{False} & \mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
 & \dots & \\
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\mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} \\
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\mathsf{Tl}\ (x{::}t) & \to & t
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
\mathsf{Subseq}\ (0{::}s)\ (0{::}t) & \to & \mathsf{Subseq}\ s\ t \\
\mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
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\mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\
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$$

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Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\ & \dots & \end{array}$$

$$
\begin{array}{rcl@{\qquad}rcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
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\mathsf{Tl}\ (x{::}t) & \to & t & \mathsf{Subseq}\ (1{::}s)\ (1{::}t) & \to & \mathsf{Subseq}\ s\ t
\end{array}
$$

Statements:

$$
\begin{array}{rcl}
\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
\mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\
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Statements:

$$\begin{array}{rcl} \mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\ \mathsf{Tl}\ (1;0;1;[]) & \to^* & 0;1;[] \\ & \dots & \\ \mathsf{Subseq}\ (0;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (0;0;1;[])\ ([]) & \to^* & \mathsf{False} \\ \mathsf{Subseq}\ (1;0;1;[])\ (0;1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (1;0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (0;1;[])\ (0;1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ (1;[])\ (1;[]) & \to^* & \mathsf{True}, \mathsf{False} \\ \mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \end{array}$$

. . .

$$
\begin{array}{rclrcl}
\mathsf{Subseq}\ []\ t & \to & \mathsf{True} & \mathsf{Subseq}\ s\ t & \to & \mathsf{Subseq}\ s\ (\mathsf{Tl}\ t) \\
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\end{array}
$$

Statements:

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Statements:

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\mathsf{Tl}\ (0;1;0;1;[]) & \to^* & 1;0;1;[] \\
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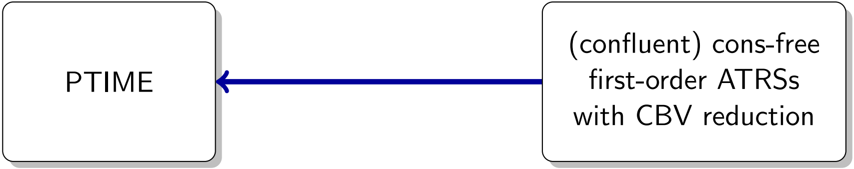
\mathsf{Subseq}\ []\ [] & \to^* & \mathsf{True}, \mathsf{False} \\
 & \dots &
\end{array}$$

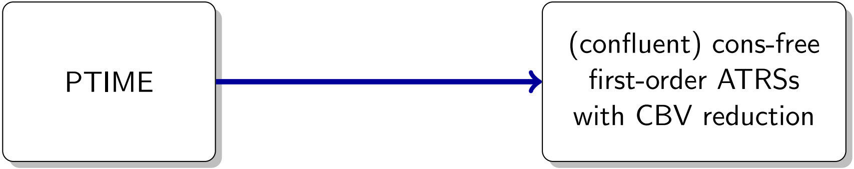
How to prove a characterisation?



For every decision problem X:

- the final state of any given Turing Machine operating in polynomial time can be found by a confluent cons-free first-order ATRS using CBV reduction
- the result of any given cons-free first-order ATRS with CBV evaluation can be found by an algorithm operating in polynomial time

How to prove a characterisation?



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- the result of any given cons-free first-order ATRS with CBV evaluation can be found by an algorithm operating in polynomial time

Claim: we can simulate any PTIME Turing Machine

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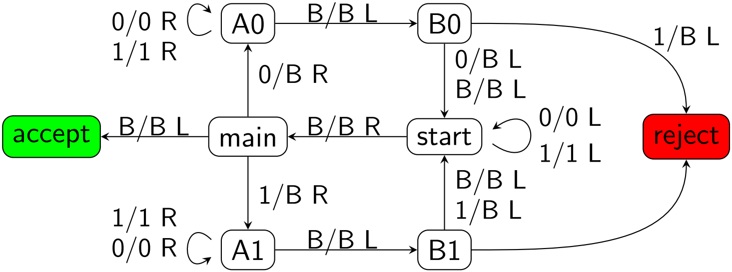
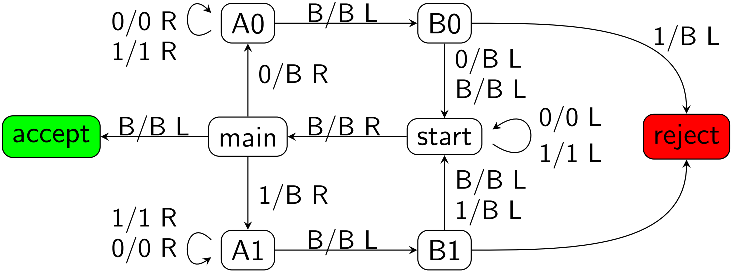
Example:



Claim: we can simulate any PTIME Turing Machine

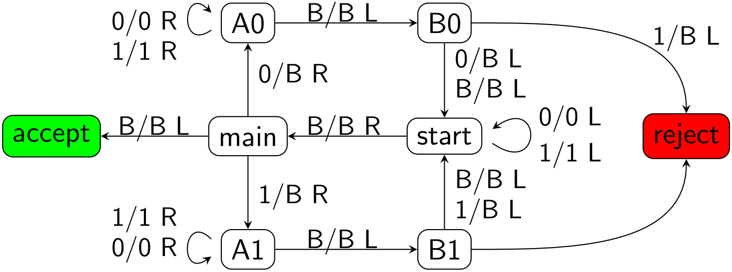
Example:



Runs in: $< 2 \cdot (n+1)^2$ steps

Claim: we can simulate any PTIME Turing Machine

Example:

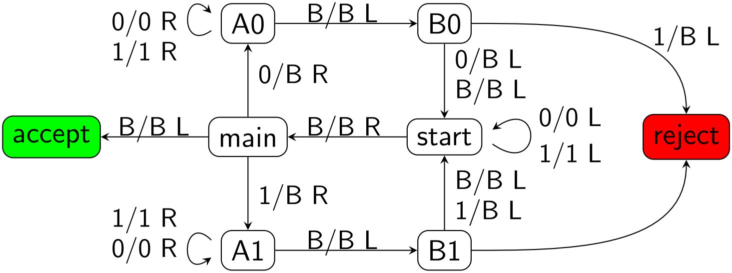


Runs in: $< 2 \cdot (n+1)^2$ steps

Transition Start 0 = (Start, 0, L)

Claim: we can simulate any PTIME Turing Machine

Example:

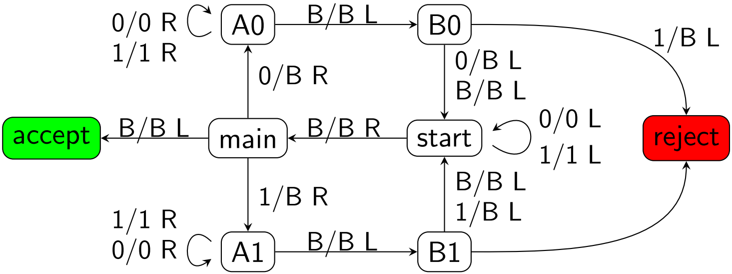


Runs in: $< 2 \cdot (n+1)^2$ steps

Transition Start 0 = (Start, 0, L)
Transition Start 1 = (Start, 1, L)
Transition Start B = (Main, B, R)

Claim: we can simulate any PTIME Turing Machine

Example:

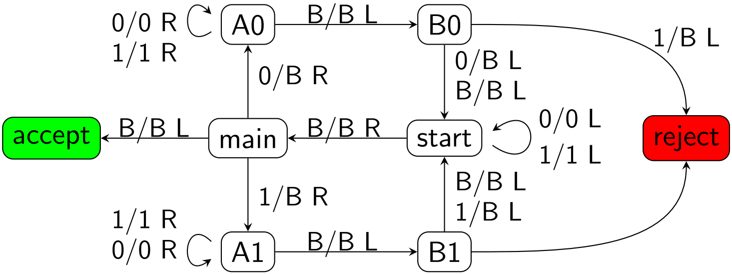


Runs in: $< 2 \cdot (n+1)^2$ steps

Transition Start 0 = (Start, 0, L)
Transition Start 1 = (Start, 1, L)
Transition Start B = (Main, B, R)
Transition Main 0 = (A0, B, R)

Claim: we can simulate any PTIME Turing Machine

Example:

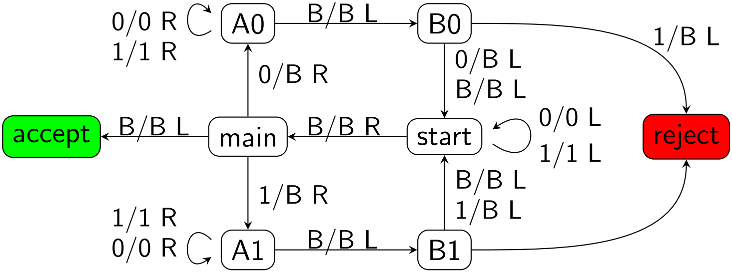


Runs in: $< 2 \cdot (n+1)^2$ steps

Transition Start 0	=	(Start, 0, L)
Transition Start 1	=	(Start, 1, L)
Transition Start B	=	(Main, B, R)
Transition Main 0	=	(A0, B, R)
Transition Accept x	=	(Accept, x, N)

Claim: we can simulate any PTIME Turing Machine

Example:



Runs in: $< 2 \cdot (n+1)^2$ steps

Transition Start 0 = (Start, 0, L)
Transition Start 1 = (Start, 1, L)
Transition Start B = (Main, B, R)
Transition Main 0 = (A0, B, R)
Transition Accept x = (Accept, x, N)

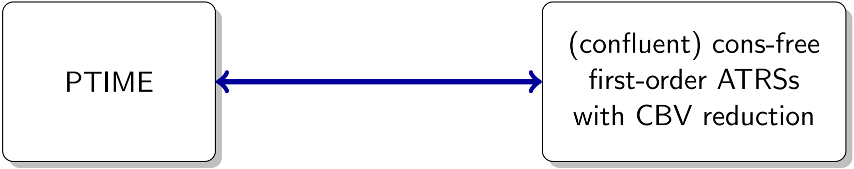
Simulation: Turing Machine running in $< 2 \cdot (n+1)^2$ steps.

$$\begin{array}{rcll} \text{Transition q r} & = & (\text{p, w, d}) & \text{for every transition } q \xrightarrow{r/w\ d} p \\ \text{Transition q } x & = & (\text{q}, x, \text{N}) & \text{for q} \in \{\text{Accept, Reject}\} \end{array}$$

Representation: $(l_1, l_2, l_3) \Longrightarrow |l_1| \cdot (n+1)^2 + |l_2| \cdot (n+1) + |l_3|$

$$\begin{array}{rcl} \text{TransAt } i\ [t] & = & \text{Transition (State } i\ [t])\ (\text{CurTape } i\ [t]) \\ \text{State } i\ [t] & = & \text{if } [t = 0] \text{ then Start} \\ & & \text{else Fst (TransAt } i\ [t-1]) \\ \text{CurTape } i\ [t] & = & \text{Tape } i\ [t]\ (\text{Pos } i\ [t]) \\ \text{Tape } i\ [t]\ [p] & = & \text{if } [t = 0] \text{ then Get } [p]\ i \text{ B} \\ & & \text{else if } [p = \text{Pos } i\ [t-1]] \text{ then} \\ & & \quad \text{Snd (TransAt } i\ [t-1]) \\ & & \text{else Tape } i\ [t-1]\ [p] \\ \text{Pos } i\ [t] & = & \text{Poshelp (Thrd (TransAt } i\ [t-1]))\ (\text{Pos } i\ [t-1]) \\ \text{PosHelp L } [p] & = & [p-1] \\ \dots & & \end{array}$$

How to prove a characterisation?



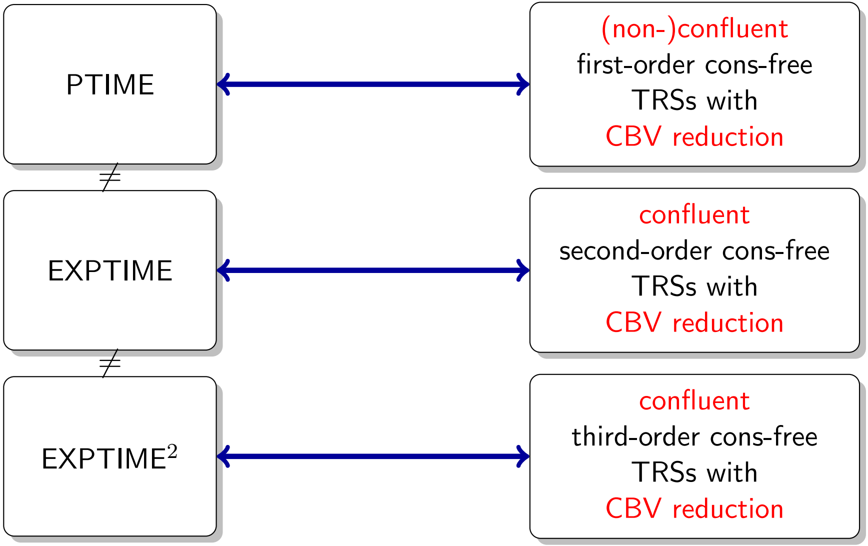
For every decision problem X:

- the result of any given cons-free first-order ATRS with CBV evaluation can be found by an algorithm operating in polynomial time
- the final state of any given Turing Machine operating in polynomial time can be found by a confluent cons-free first-order ATRS using CBV reduction

HIGHER-ORDER CHARACTERISATIONS

Overview

1. cons-free applicative rewriting
   (what is this "cons-freeness" and how do we use it?)
2. characterisations with first-order cons-free innermost rewriting
   (the general idea)
3. characterisations with higher-order cons-free innermost rewriting
   (where it starts to get interesting)
4. characterisations using non-innermost cons-free rewriting
   (where it really gets interesting)


PTIME
≠
EXPTIME
≠
EXPTIME^2
(non-)confluent
first-order cons-free
TRSs with
CBV reduction
confluent
second-order cons-free
TRSs with
CBV reduction
confluent
third-order cons-free
TRSs with
CBV reduction

Simulation: Turing Machine running in $< 2 \cdot (n+1)^2$ steps.

Transition q r = (p, w, d) for every transition $q \overset{r/w\ d}{\Longrightarrow} p$
Transition q x = (q, x, N) for q ∈ {Accept, Reject}

Representation: $(l_1, l_2, l_3) \Longrightarrow |l_1| \cdot (n+1)^2 + |l_2| \cdot (n+1) + |l_3|$

TransAt i $[t]$ = Transition (State i $[t]$) (CurTape i $[t]$)
State i $[t]$ = if $[t = 0]$ then Start
else Fst (TransAt i $[t-1]$)
CurTape i $[t]$ = Tape i $[t]$ (Pos i $[t]$)
Tape i $[t]$ $[p]$ = if $[t = 0]$ then Get $[p]$ i B
else if $[p =$ Pos i $[t-1]]$ then
Snd (TransAt i $[t-1]$)
else Tape i $[t-1]$ $[p]$
Pos i $[t]$ = Poshelp (Thrd (TransAt i $[t-1]$)) (Pos i $[t-1]$)
PosHelp L $[p]$ = $[p-1]$
...

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- represent numbers $< A \cdot (n+1)^B$ by tuples $(l_1, \dots, l_B, l_{B+1})$
- represent numbers $< 2^{A \cdot (n+1)^B}$ as values in list $\Rightarrow$ bool
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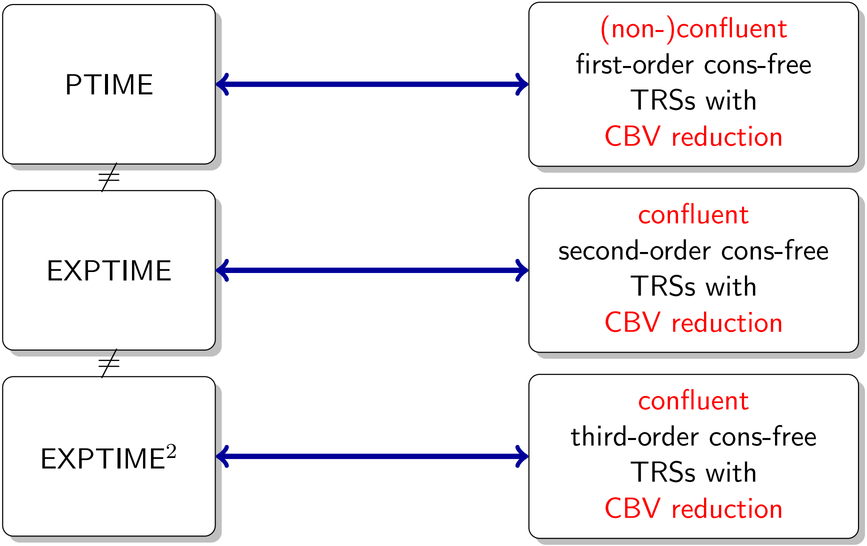
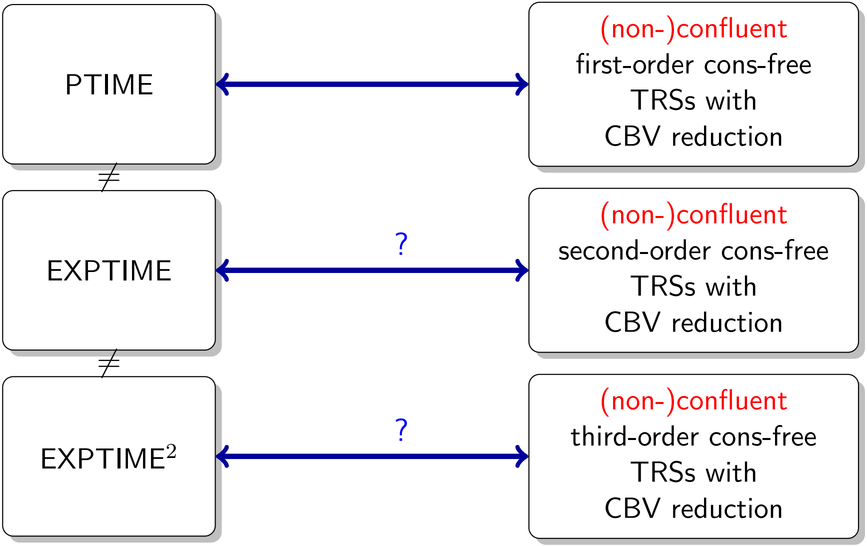
Conclusion:

EXPTIME^K $\longrightarrow$ (confluent) cons-free $(K+1)$-order ATRSs with CBV reduction

In EXPTIME^K $\Rightarrow$ accepted by an orthogonal cons-free $(K+1)$order ATRS!




PTIME
(non-)confluent
first-order cons-free
TRSs with
CBV reduction
≠
EXPTIME
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(non-)confluent
second-order cons-free
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(non-)confluent
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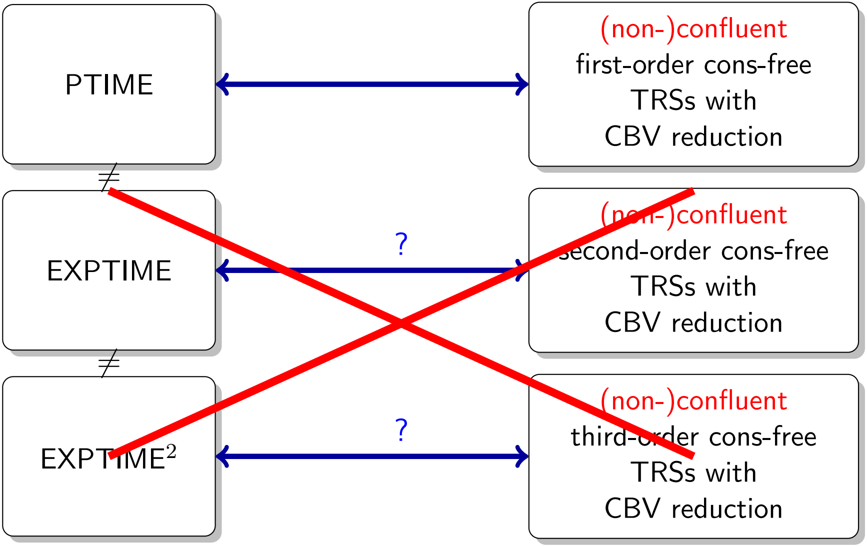
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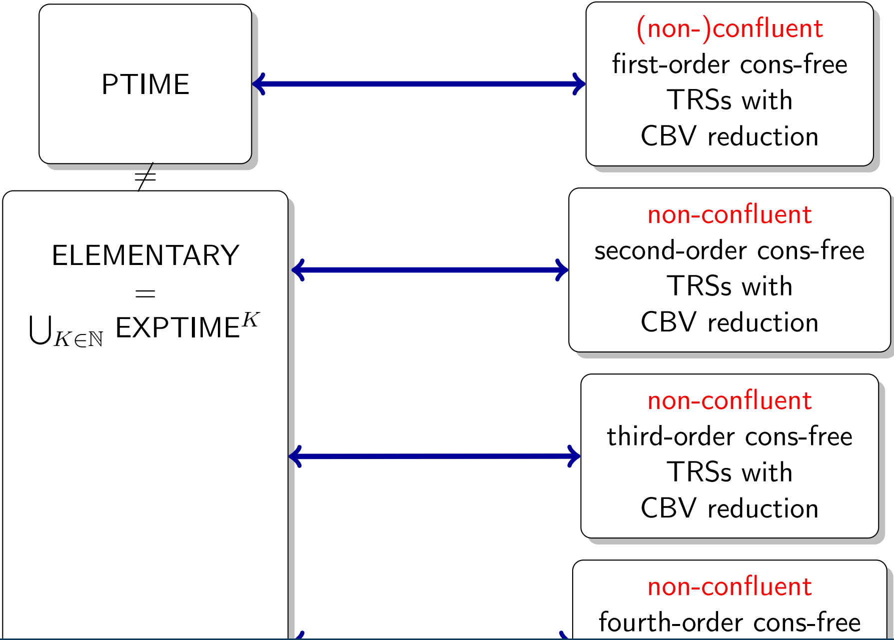
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fourth-order cons-free

Conclusion:

$\Longrightarrow$ Functional variables + non-determinism + CBV =

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$\Longrightarrow$ Why?

Key insight

- In a confluent ATRS, an element of $\sigma \Rightarrow \tau$ represents a function from T_σ to T_τ.
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- In a confluent ATRS, an element of $\sigma \Rightarrow \tau$ represents a function from T_σ to T_τ.
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- In a non-confluent innermost ATRS, an element of $\sigma \Rightarrow \tau$ represents a function from T_σ to $\mathcal{P}(T_\tau)$.
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Logical Conclusion:

if we can count up to $\mathtt{exp}_2^K(n)$ for any K
then we can simulate any TM running in time bounded by some $\mathtt{exp}_2^K(n)$
so we can handle all problems in ELEMENTARY

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$$
\begin{array}{lcl}
\mathsf{Set1}\ n\ f\ \mathsf{True} & = & \mathsf{choose}\ n\ (f\ \mathsf{True}) \\
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Bit vector 10110**:**

Set1 "1" (Set0 "2" (Set1 "3" (Set1 "4" (Set0 "5" (Const "0")))))

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  use non-determinism!

$$
\begin{array}{lcl}
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 & & \text{else if } (\text{Equal } (f \text{ False})\ n) \text{ then False} \\
 & & \text{else Crash}
\end{array}
$$

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PTIME

↔

(non-)confluent
first-order cons-free
TRSs with
CBV reduction

$\neq$

ELEMENTARY
$=$
$\bigcup_{K \in \mathbb{N}}$ EXPTIME^K

↔

non-confluent
second-order cons-free
TRSs with
CBV reduction

↔

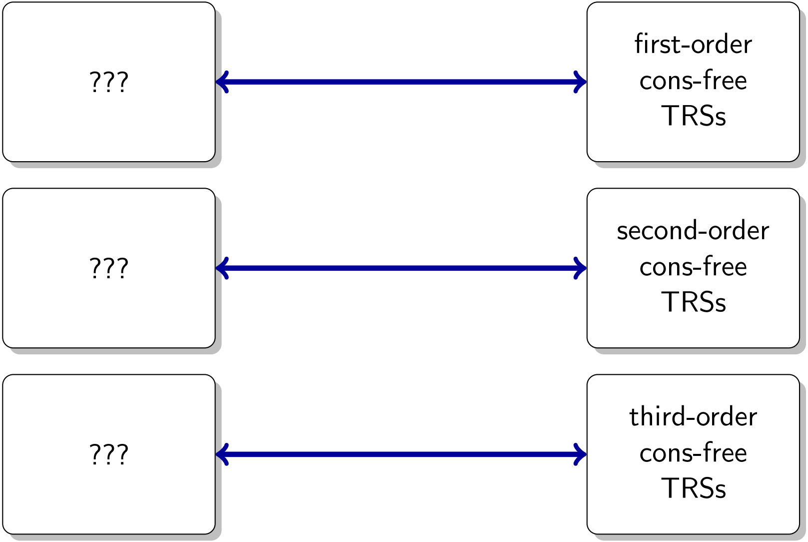
non-confluent
third-order cons-free
TRSs with
CBV reduction

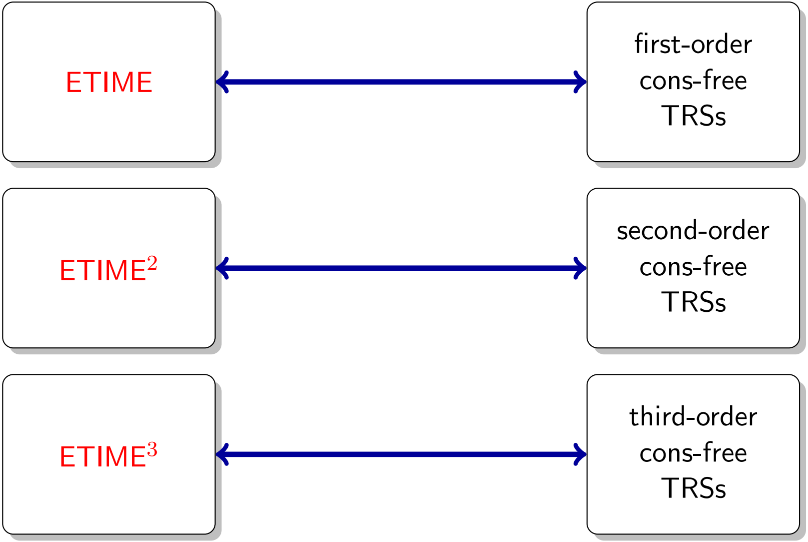
non-confluent
fourth-order cons-free

USING ARBITRARY EVALUATION STRATEGIES

Overview

1. cons-free applicative rewriting
   (what is this "cons-freeness" and how do we use it?)
2. characterisations with first-order cons-free innermost rewriting
   (the general idea)
3. characterisations with higher-order cons-free innermost rewriting
   (where it starts to get interesting)
4. characterisations using non-innermost cons-free rewriting
   (where it really gets interesting)


???
first-order cons-free TRSs
???
second-order cons-free TRSs
???
third-order cons-free TRSs


ETIME
first-order
cons-free
TRSs
ETIME^2
second-order
cons-free
TRSs
ETIME^3
third-order
cons-free
TRSs

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TO CONCLUDE

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- proofs are robust if determinism is preserved

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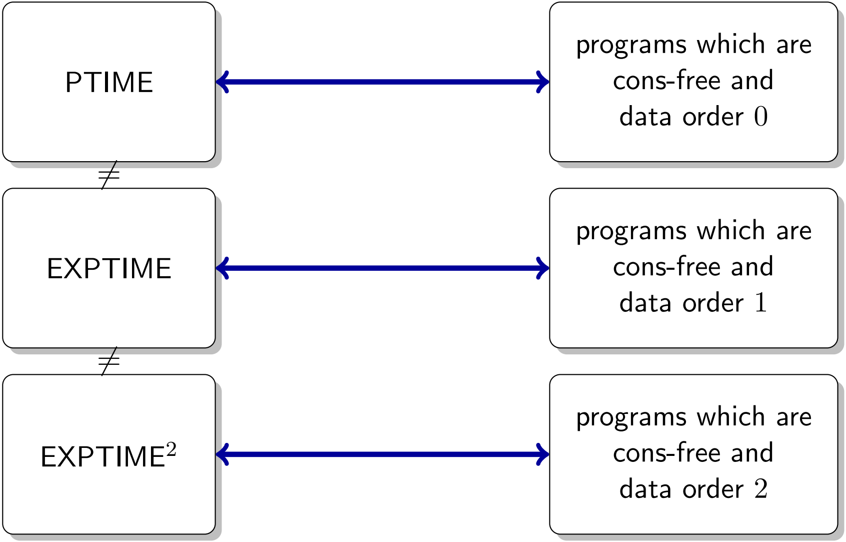
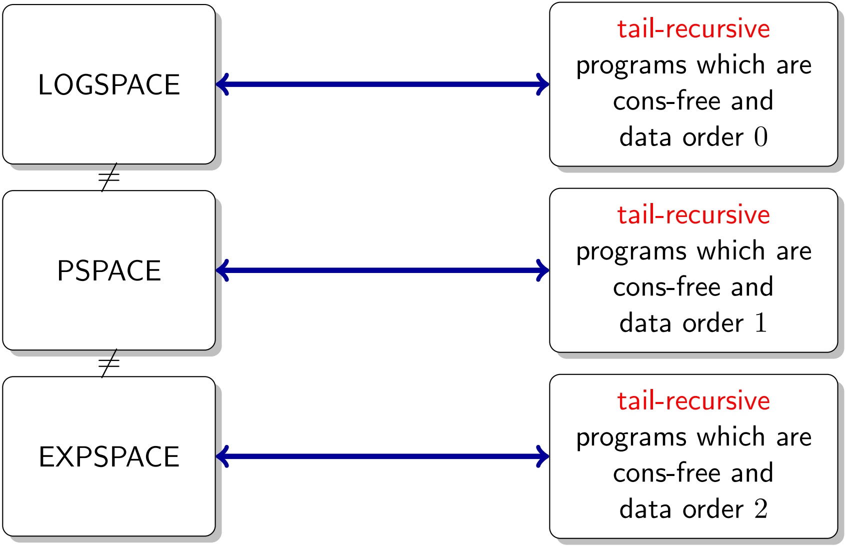
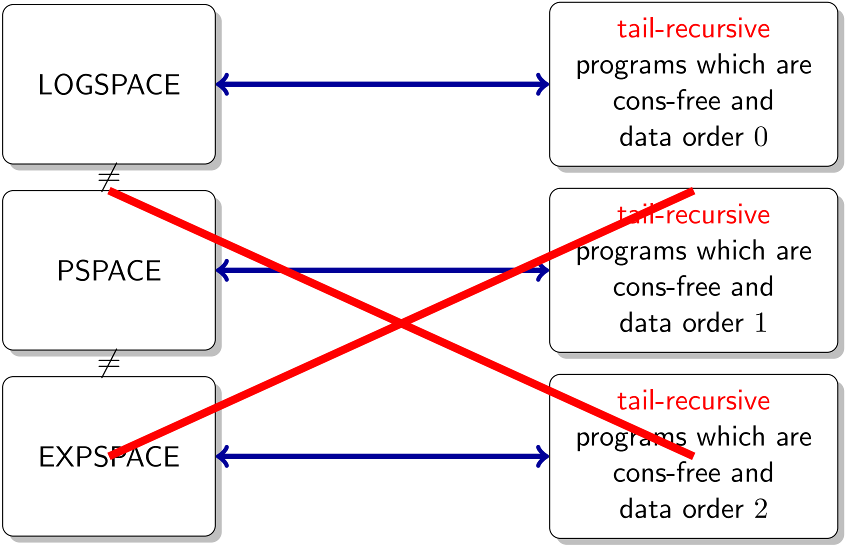
Questions?

TAIL-RECURSIVE CONS-FREE TERM REWRITING


PTIME
programs which are cons-free and data order 0
$\neq$
EXPTIME
programs which are cons-free and data order 1
$\neq$
EXPTIME^2
programs which are cons-free and data order 2


LOGSPACE
tail-recursive programs which are cons-free and data order 0
≠
PSPACE
tail-recursive programs which are cons-free and data order 1
≠
EXPSPACE
tail-recursive programs which are cons-free and data order 2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EXPSPACE
tail-recursive programs which are cons-free and data order 2

What is tail-recursion?

What is tail-recursion?

In first-order rewriting:

- impose an ordering on the function symbols $\mathsf{F} \succeq \mathsf{G}$ (always $\mathsf{F} \succ \mathsf{C}$)
- for all rules $\mathsf{F}\ \ell_1 \cdots \ell_k \to r$:
  - if $r = \mathsf{G}\ r_1 \cdots r_n$, then $\mathsf{F} \succeq \mathsf{G}$
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- Should we count calls $\mathsf{G}\ x$ with insufficient arguments? $\mathsf{F}(\mathsf{S}(n)) \to \mathsf{G}\ \mathsf{F}$ with $\mathsf{F} \succ \mathsf{G}$?