On Confluence of Innermost Terminating Term Rewriting Systems

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joint work with
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This talk

- Background:
  - Open problems on confluence for innermost terminating TRSs
- Tackling the open problem in negative direction
  - A simple proof of undecidability of ground confluence for terminating TRSs
- Sufficient conditions of confluence for innermost terminating TRSs

Confluence for terminating TRSs

- **Well-known result of [KB'70]:**
  for $\mathrm{SN}(\rightarrow_{\mathcal{R}})$
  $\mathrm{CP}_{\mathcal{R}} \subseteq \xrightarrow{*}_{\mathcal{R}} \cdot \xleftarrow{*}_{\mathcal{R}}$ **iff** $\mathrm{CR}(\rightarrow_{\mathcal{R}})$
- **Notations:**
  $\mathrm{SN}(\rightarrow)$: $\rightarrow$ **is terminating**
  $\mathrm{CR}(\rightarrow)$: $\rightarrow$ **is Church-Rosser** ($\xleftrightarrow{*} \subseteq \xrightarrow{*} \cdot \xleftarrow{*}$)
  $\mathrm{Conf}(\rightarrow)$: $\rightarrow$ **is confluent** ($\xleftarrow{*} \cdot \xrightarrow{*} \subseteq \xrightarrow{*} \cdot \xleftarrow{*}$)
  $\mathrm{NF}(\rightarrow)$: **the set of normal forms**
  $\mathrm{CP}_{\mathcal{R}}$: **the set of critical pairs of** $\mathcal{R}$

Critical pairs

- **Ex.**

  $\mathcal{R} = \{h(k(x)) \to f(x),\ k(c) \to d\}$

  $h(d) \leftarrow_{\mathcal{R}} h(k(c)) \to_{\mathcal{R}} f(c)$

  $\mathsf{CP}_{\mathcal{R}} = \{(h(d), f(c))\}$

- **Formally,**

  for $\ell \to r, \ell' \to r' \in \mathcal{R}$ **(no common variables),**
  let $\theta = \mathsf{mgu}(\ell|_p, \ell')$ **for some pos.** p**,**
  i.e. $\ell\theta[r'\theta]_p \leftarrow_{\mathcal{R}} \ell\theta[\ell'\theta]_p = \ell\theta \to_{\mathcal{R}} r\theta$
  Then $(\ell\theta[r'\theta]_p, r\theta)$ **is a critical pair**

Confluence for terminating TRSs

- **Well-known result of** [KB'70]**:**
  for $\mathsf{SN}(\rightarrow_{\mathcal{R}})$
  $\mathsf{CP}_{\mathcal{R}} \subseteq \xrightarrow{*}_{\mathcal{R}} \cdot \xleftarrow{*}_{\mathcal{R}}$ **iff** $\mathsf{CR}(\rightarrow_{\mathcal{R}})$
- **How about on innermost terminating TRSs?**
  - **Innermost rewrite step** $\underset{i}{\rightarrow}$ **rewrites a redex having no redex strictly below**

Confluence for innermost terminating TRSs

- **An open problem [Ohlebusch'01]:**
  for $\mathsf{SN}(\xrightarrow[\mathsf{i}]{}_{\mathcal{R}})$
  $$\mathsf{CP}_{\mathcal{R}} \subseteq \xrightarrow[\mathsf{i}]{*}_{\mathcal{R}} \cdot \xleftarrow[\mathsf{i}]{*}_{\mathcal{R}} \text{ iff } \mathsf{CR}(\to_{\mathcal{R}})$$
- **A partial result for the problem [Gramlich'95]**

  Th. The claim holds if $\mathsf{OV}(\mathcal{R})$
  Proof: by showing $\mathsf{SN}(\to_{\mathcal{R}})$
- **$\mathcal{R}$ is overlay $\mathsf{OV}(\mathcal{R})$: CPs are all produced from root-overlaps**

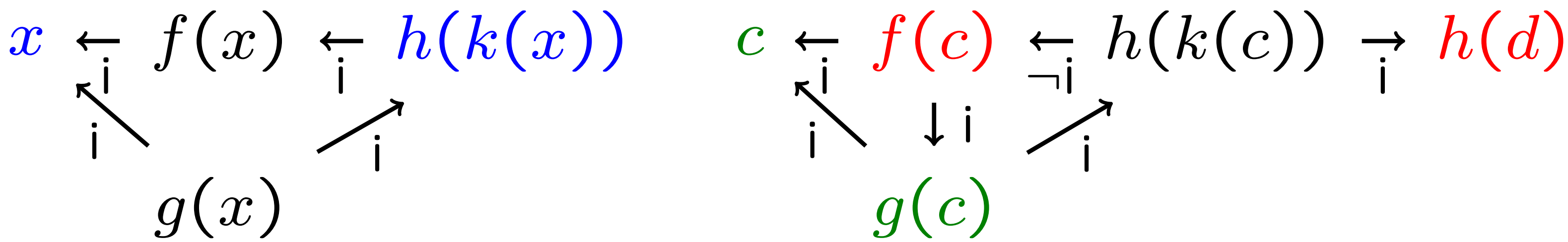
The open prob. is negatively solved

- **Counter example [presented at IWC'16]:**

$$\mathcal{R} = \{f(x) \to x,\ h(k(x)) \to f(x),\ k(c) \to d,\ g(x) \to x,\ f(c) \to g(c),\ g(x) \to h(k(x))\}$$

$$\mathrm{CP}_{\mathcal{R}} = \{(x, h(k(x))),\ (h(d), f(c)),\ (c, g(c))\}$$

- $\mathrm{SN}(\xrightarrow[\mathsf{i}]{}_{\mathcal{R}})$ **and** $\mathrm{CP}_{\mathcal{R}} \subseteq \xrightarrow[\mathsf{i}]{*}_{\mathcal{R}} \cdot \xleftarrow[\mathsf{i}]{*}_{\mathcal{R}}$**, but is not confluent**



Characterization of confluence [presented at IWC'16]

- **Instances of CPs by normalized substitutions**
  iCP : $\{(u\sigma, v\sigma) \mid (u,v) \in \mathsf{CP},\ \sigma$: **normalized subst.**$\}$
  Note that iCP $\supseteq$ CP
- **Th. For** $\mathsf{SN}(\underset{\mathsf{i}}{\rightarrow})$,

$$\mathsf{iCP} \subseteq \underset{\mathsf{i}}{\overset{*}{\rightarrow}} \cdot \underset{\mathsf{i}}{\overset{*}{\leftarrow}} \quad \textbf{iff} \quad \mathsf{CR}(\rightarrow)$$

- **New open problem**
  Is confluence of innermost terminating TRSs decidable?

This talk

- Background:
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Tackling the new open problem

- Tried to show undecidability of confluence in vain, and have shown undecidability of ground confluence for terminating TRSs [Kapur'87])
- Normal technique to show undecidability: reducing an undecidable problem (here PCP) into confluence problem
  We adopt the policy that
  an instance has a solution iff non-confluent
  - Difficulties: necessary to design innermost terminating TRSs

PCP (Post's correspondence problem)

- **Instance: a set P of pair of strings**
  Ex. $P = \{(aa, a),\ (b, ab)\}$
- **Answer: whether there exist non-empty sequence $(u_i, v_i) \in P$ ($1 \leq i \leq n$) s.t.**
  $u_1 \cdots u_n = v_1 \cdots v_n$ **(called solution)**
  For ex. above, it is YES because sequence $(aa, a), (b, ab)$ gives a solution aab
- **Known as undecidable whether an instance has a solution or not**

TRS construction for confluence

- **Unary representation $a(a(b(e)))$ of string aab (abbreviated as $aab(e)$)**
- **For an instance $\{(aa,a),\ (b,ab)\}$,**

$$
\begin{aligned}
R = \{ & f(x,x,y) \to c, \\
& f(aa(x), a(y), 1(z)) \leftrightarrow f(x,y,z), \\
& f(aa(e), a(e), 1(e)) \to c', \\
& f(b(x), ab(y), 2(z)) \leftrightarrow f(x,y,z), \\
& f(b(e), ab(e), 2(e)) \to c' \quad \}
\end{aligned}
$$

$$
c \underset{\mathsf{i}}{\leftarrow} f\left(\begin{array}{c} aab(e) \\ aab(e) \\ 1(2(e)) \end{array}\right) \underset{\mathsf{i}}{\longleftrightarrow} f\left(\begin{array}{c} b(e) \\ ab(e) \\ 2(e) \end{array}\right) \underset{\mathsf{i}}{\rightarrow} c'
$$

TRS construction for confluence

- **For an instance** $\{(aa, a), (b, ab)\}$,

$$
\begin{aligned}
R = \{ & f(x, x, y) \to c, \\
& f(aa(x), a(y), 1(z)) \leftrightarrow f(x, y, z), \\
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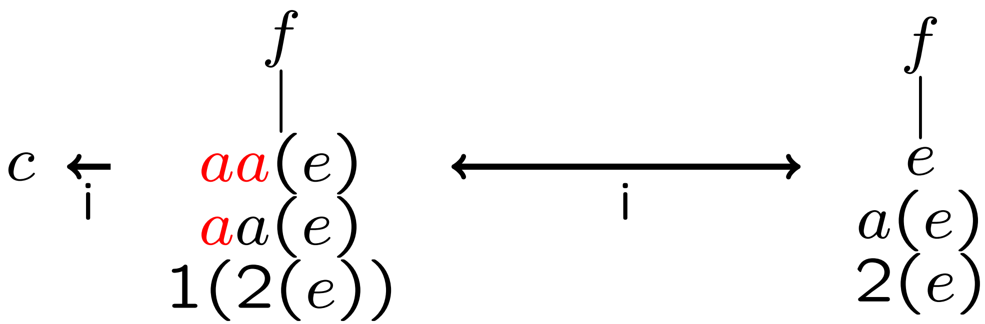
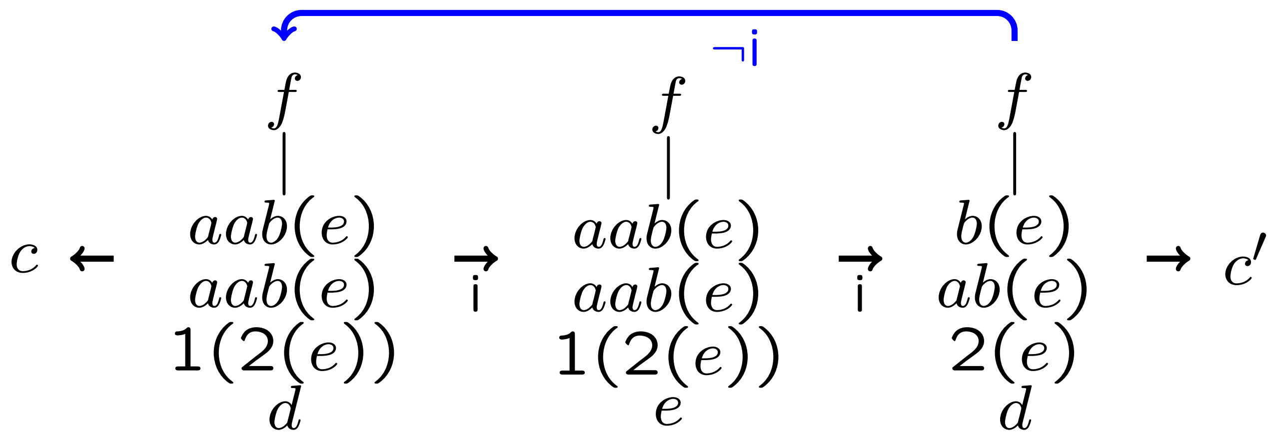
& f(b(e), ab(e), 2(e)) \to c' \quad \}
\end{aligned}
$$



Innermost-terminating TRS constr.

- **Idea: introduce an argument that control innermost or not**

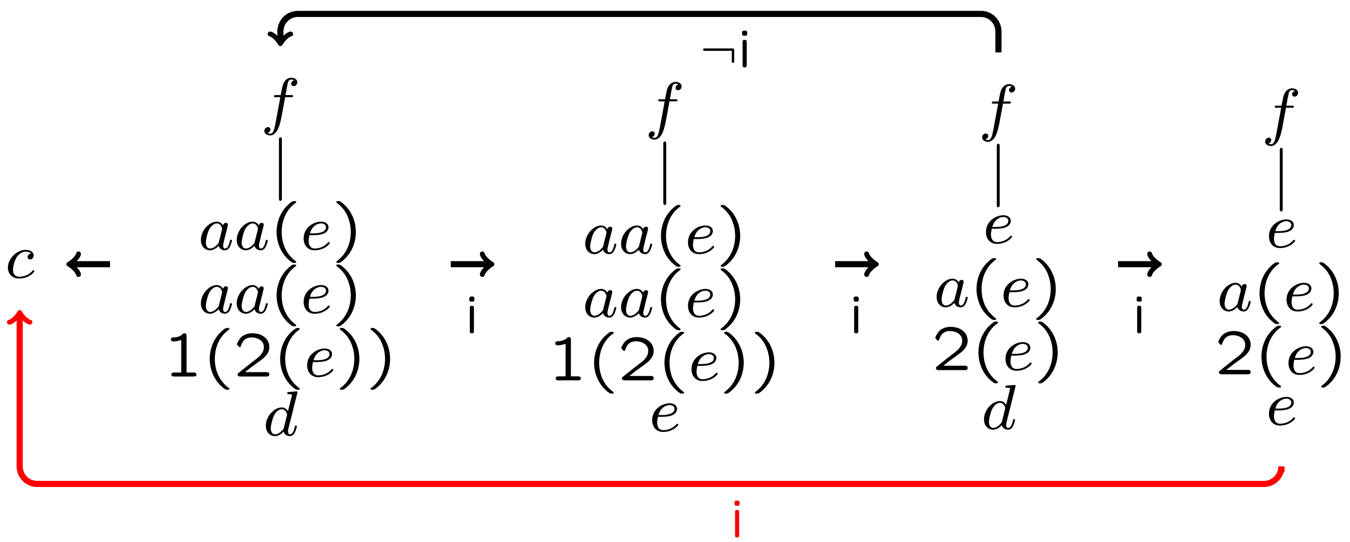
$$
\begin{aligned}
R = \{ & d \to e, \;\; f(x,x,y,z) \to c, \\
& f(aa(x), a(y), 1(z), d) \leftarrow f(x,y,z,d), \\
& f(aa(x), a(y), 1(z), e) \to f(x,y,z,d), \\
& f(aa(e), a(e), e, z') \to c', \quad \cdots \quad \}
\end{aligned}
$$



Innermost-terminating TRS constr.

- **Rules avoiding unexpected normal forms caused by non-solution (complement rules)**

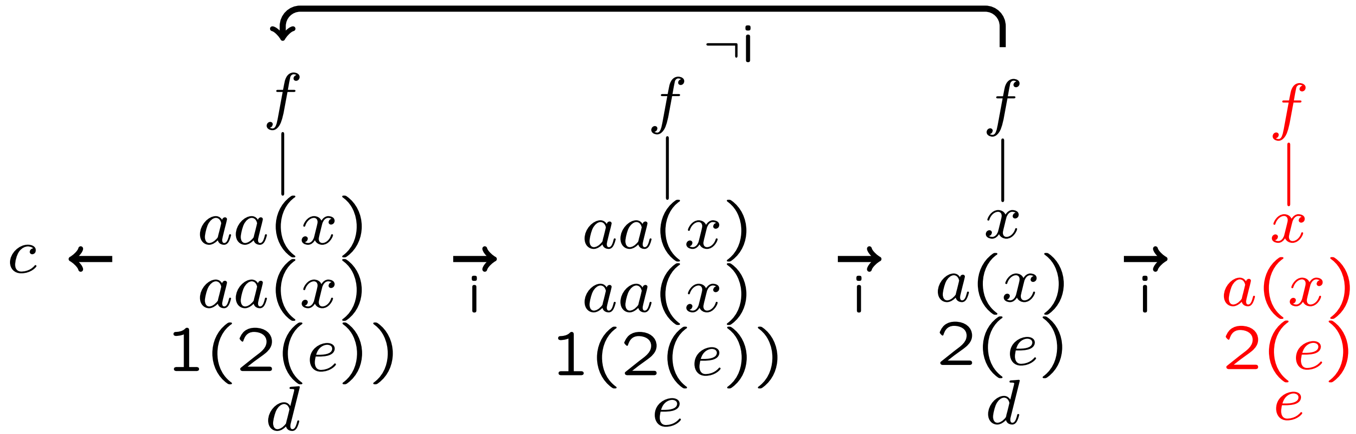
$$R = \{d \to e,\ f(x,x,y,z') \to c,$$
$$f(aa(x), a(y), 1(z), e) \to f(x,y,z,d),$$
$$f(b(x), ab(y), 2(z), e) \to f(x,y,z,d),$$
$$f(e,y,z,z') \to c,\ f(b(x), y, 1(z), z') \to c,, \ \cdots \ \}$$



Problem on non-ground terms

- **Unexpected normal forms cannot be removed due to variables. Construction failed.**

$$R = \{d \to e,\ f(x,x,y,z') \to c,$$
$$f(aa(x), a(y), 1(z), e) \to f(x,y,z,d),$$
$$f(b(x), ab(y), 2(z), e) \to f(x,y,z,d),$$
$$f(e,y,z,z') \to c,\ f(b(x), y, 1(z), z') \to c,\ \cdots\ \}$$

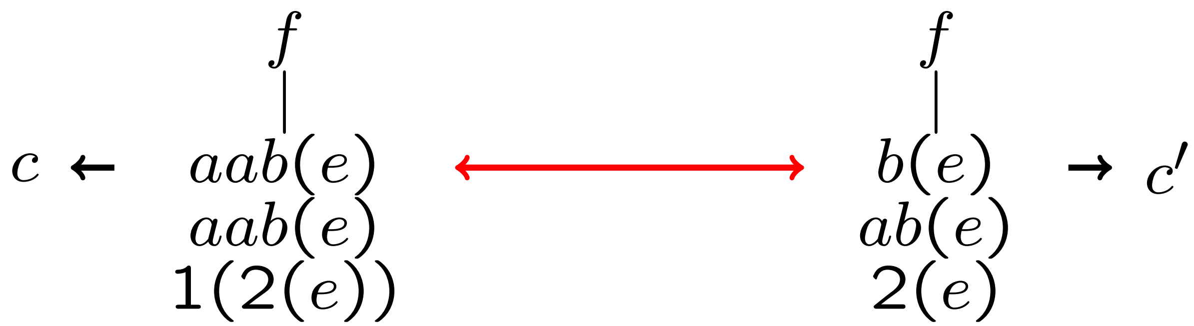


- **Obtained undecidability of ground confluence**

Terminating TRS construction for grand confluence

- **NON-terminating const. (already shown)**

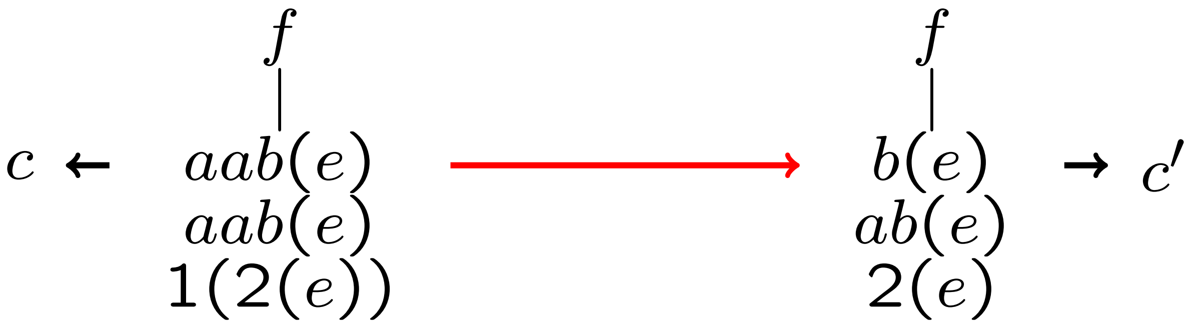
$$
\begin{aligned}
R = \{ & f(x,x,z) \to c, \\
& f(aa(x), a(y), 1(z)) \leftrightarrow f(x,y,z), \\
& f(aa(e), a(e), 1(e)) \to c', \\
& f(b(x), ab(y), 2(z)) \leftrightarrow f(x,y,z), \\
& f(b(e), ab(e), 2(e)) \to c' \quad \}
\end{aligned}
$$



Terminating TRS construction for grand confluence

- **terminating const.**

$$
\begin{aligned}
R = \{ & f(x,x,z) \to c, \\
& f(aa(x), a(y), 1(z)) \to f(x,y,z), \\
& f(aa(e), a(e), 1(e)) \to c', \\
& f(b(x), ab(y), 2(z)) \to f(x,y,z), \\
& f(b(e), ab(e), 2(e)) \to c' \quad \} \cup \textbf{complement rules}
\end{aligned}
$$



Terminating TRS construction for grand confluence

- **terminating const.**

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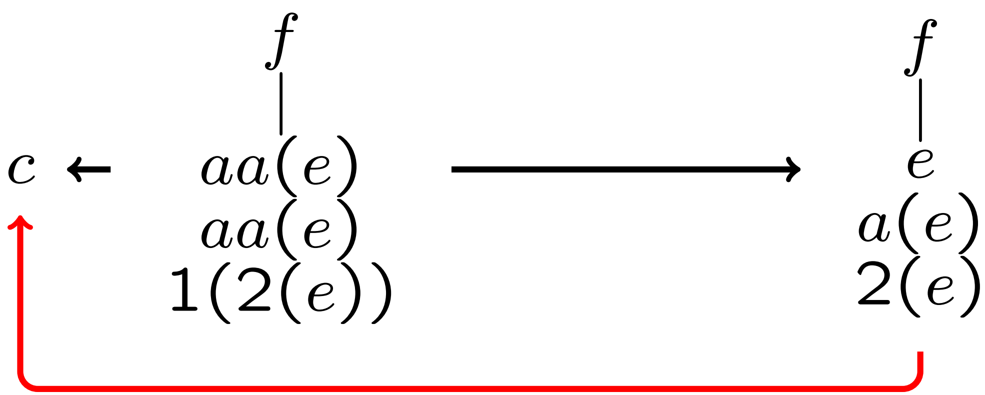

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- **Background:**
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  - **A simple proof of undecidability of ground confluence for terminating TRSs**
- **Sufficient conditions of confluence for innermost terminating TRSs**

Characterization of confluence [presented at IWC'16]

- **Th. For** $\mathsf{SN}(\underset{\mathsf{i}}{\to})$**,**

  $$\mathsf{iCP} \subseteq \overset{*}{\underset{\mathsf{i}}{\to}} \cdot \overset{*}{\underset{\mathsf{i}}{\leftarrow}} \quad \textbf{iff} \quad \mathsf{CR}(\to)$$

  where

  $\mathsf{iCP} = \{(u\sigma, v\sigma) \mid (u, v) \in \mathsf{CP},\ \sigma : \textbf{normal. subst.}\}$

  Note that $\mathsf{iCP} \supseteq \mathsf{CP}$

Sufficient conditions

- **Ground innermost rewrite step $s \mathrel{\underset{\mathsf{gi}}{\to}} t$:**

  $s \mathrel{\underset{\mathsf{i}}{\to}} t$ **and the redex is a ground term**

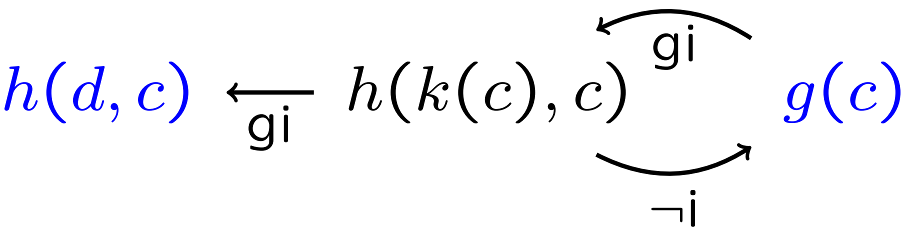
- $\underset{\mathsf{gi}}{\to}$ **rewrite sequences are stable under substitutions**

- **Th.[presented at IWC'16] For** $\mathsf{SN}(\underset{\mathsf{i}}{\to})$,

  $\mathsf{CP} \subseteq \overset{*}{\underset{\mathsf{gi}}{\to}} \cdot \overset{*}{\underset{\mathsf{gi}}{\leftarrow}} \implies \mathsf{CR}(\to)$

Sufficient conditions

- **Bidirectional parallel step** $s \mathrel{\leftrightarrow\!\!\!\!\!\!\!\!\!+\!\!\!\!+} t$**:**
  $s = s[s_1, \ldots, s_n]$, $t = s[t_1, \ldots, t_n]$, **and** $s_i \leftrightarrow t_i$
- **Th. For** $\mathsf{SN}(\underset{\mathsf{i}}{\to})$**,**
  $$\mathsf{CP} \subseteq \xrightarrow[\mathsf{gi}]{*} \cdot \mathrel{\leftrightarrow\!\!\!\!\!\!\!\!\!+\!\!\!\!+} \cdot \xleftarrow[\mathsf{gi}]{*} \implies \mathsf{CR}(\to)$$
  Proven by Lem. in the next slide
- **An example:**
  $$R = \{k(c) \to d,\ h(k(x), x) \to g(x),\ g(c) \to h(k(c), c)\}$$
  $$\mathsf{CP}_{\mathcal{R}} = \{(h(d, c), g(c))\}$$



Sufficient conditions

- **Lem. For** SN($\underset{\mathsf{i}}{\to}$),

  iCP $\subseteq \overset{*}{\underset{\mathsf{i}}{\to}} \cdot \leftrightarrow\!\!\!\!\!\!\!\!\;\,\vdash\!\!\dashv \cdot \overset{*}{\underset{\mathsf{i}}{\leftarrow}}$ **iff** CR($\to$)

  Proof (⇒): Show that $s \leftrightarrow\!\!\!\!\!\!\!\!\;\,\vdash\!\!\dashv t$ **implies** $s \overset{*}{\underset{\mathsf{i}}{\to}} \cdot \overset{*}{\underset{\mathsf{i}}{\leftarrow}} t$ **by Noetherian induction on** $\{s, t\}$ **wrt multiset extension of** $\underset{\mathsf{i}}{\to} \cup \rhd$ **(**$\rhd$**: proper subterm rel.)**

  Case $s = f(s_1, \ldots, s_n) \leftrightarrow\!\!\!\!\!\!\!\!\;\,\vdash\!\!\dashv^{\varepsilon<} f(t_1, \ldots, t_n) = t$

  $s \rhd s_i$, $t \rhd t_i$ **and** $s_i \leftrightarrow\!\!\!\!\!\!\!\!\;\,\vdash\!\!\dashv t_i$ **for each** i

  By IH $s_i \overset{*}{\underset{\mathsf{i}}{\to}} \cdot \overset{*}{\underset{\mathsf{i}}{\leftarrow}} t_i$

  Hence, $s \overset{*}{\underset{\mathsf{i}}{\to}} \cdot \overset{*}{\underset{\mathsf{i}}{\leftarrow}} t$

  Case $s = \ell\sigma \underset{\mathsf{i}}{\to} r\sigma = t$: **it is trivial**

Sufficient conditions

- **Lem. For** SN($\xrightarrow[\mathsf{i}]{}$),
  iCP $\subseteq \xrightarrow[\mathsf{i}]{*} \cdot \leftrightarrow\!\!\!\!\!\!\mid\!\mid\!\!\!\!\!\!\rightarrow \cdot \xleftarrow[\mathsf{i}]{*}$ **iff** CR($\rightarrow$)

  Proof ($\Rightarrow$): Show that $s \leftrightarrow\!\!\!\!\!\!\mid\!\mid\!\!\!\!\!\!\rightarrow t$ implies $s \xrightarrow[\mathsf{i}]{*} \cdot \xleftarrow[\mathsf{i}]{*} t$
  by Noetherian induction on $\{s, t\}$ wrt multiset extension of $\xrightarrow[\mathsf{i}]{} \cup \rhd$ ($\rhd$: proper subterm rel.)

  Case $s = \ell\sigma \xrightarrow[\neg\mathsf{i}]{} r\sigma = t$:

  - **If σ is not normalized then $s \xrightarrow[\mathsf{i}]{+} \ell\sigma' \rightarrow r\sigma' \xleftarrow[\mathsf{i}]{*} t$**
  - **If σ is normalized, then a CP (u, v) exists s.t. $s \xrightarrow[\mathsf{i}]{} u\theta \xrightarrow[\mathsf{i}]{*} s' \leftrightarrow\!\!\!\!\!\!\mid\!\mid\!\!\!\!\!\!\rightarrow t' \xleftarrow[\mathsf{i}]{*} v\theta = t$**

  The case follows from IH

Another condition

- **Left-stable rule** $\ell \to r$: $\ell\sigma \underset{\mathsf{i}}{\to} r\sigma$ **for any normalized substitution** σ

  $\to_{\mathsf{ls}}$: **Rewrite step by a left-stable rule**

- **Rewrite sequences of** $\underset{\mathsf{i}}{\to}_{\mathsf{ls}}$ **are, however, not stable under normalized substitution**

  $R = \{f(x) \to g(h(x)),\ h(a) \to b\}$

  $$f(f(x)) \underset{\mathsf{i}}{\to}_{\mathsf{ls}} f(g(h(x))) \underset{\mathsf{i}}{\to}_{\mathsf{ls}} g(h(g(h(x))))$$

  $$f(f(a)) \underset{\mathsf{i}}{\to}_{\mathsf{ls}} f(g(h(a))) \underset{\neg\mathsf{i}}{\to}_{\mathsf{ls}} g(h(g(h(a))))$$

- **Nevertheless, we have a theorem**

  Th. For $\mathsf{SN}(\underset{\mathsf{i}}{\to})$**,**

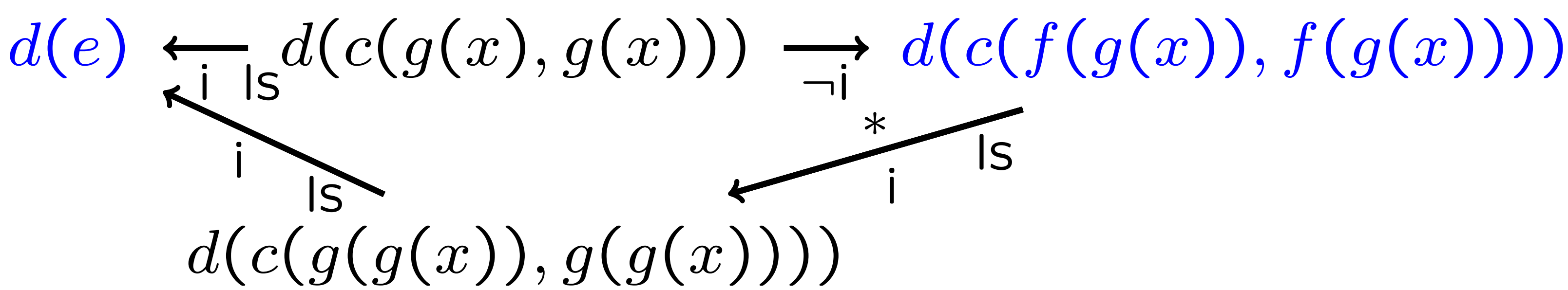
  $$\mathsf{CP} \subseteq \overset{*}{\underset{\mathsf{i}}{\to}}_{\mathsf{ls}} \cdot \overset{*}{\underset{\mathsf{i}}{\leftarrow}}_{\mathsf{ls}} \implies \mathsf{CR}(\to)$$

Another condition

- **An example:**

$$R = \{c(g(x), g(x)) \to_{\mathsf{ls}} e,\ d(c(x,x)) \to d(c(f(x), f(x))),\ f(x) \to_{\mathsf{ls}} g(x),\ f(d(e)) \to_{\mathsf{ls}} g(d(e))\}$$

$$\mathsf{CP}_{\mathcal{R}} = \{(d(e), d(c(f(g(x)), f(g(x))))), (g(d(e)), g(d(e)))\}$$



None of ACP, CSI, Saigawa (2016 CoCo versions) could prove this

Another condition

- **Left-stability is decidable from Lem. below (similar to critical pairs)**
- **Lem.** $\ell \to r \in \mathcal{R}$ **is NOT left-stable iff**
  there exist $\ell' \to r' \in \mathcal{R}$ **and** p **(**$\neq \varepsilon$**) s.t.**
  $x\theta$ **is normalized for all** $x \in \mathsf{Var}(\ell)$
  where $\theta = \mathsf{mgu}(\ell|_p, \ell')$.

Notions used in proving the theorem

- **Priority TRS** [Baeten'89]: **to provide an inner-most reduction** $\underset{i}{\Rightarrow}$ **satisfying that**
  - $\underset{i}{\Rightarrow} \subseteq \underset{i}{\rightarrow}$,
  - $NF_{\underset{i}{\Rightarrow}} = NF_{\underset{i}{\rightarrow}}$, **and**
  - CR($\underset{i}{\Rightarrow}$)

Notions used in proving the theorem

- **Basic reduction: a rewrite version of basic narrowing** [Hullot'80]
  - **A rewrite sequence is basic if redexes substituted under variable are locked so that never reduced**

$$R = \{f(x) \to g(h(x), x),\ h(a) \to b,\ a \to c\}$$

$f(a) \to g(h(a), a) \to g(b, a)$ **is basic, but**
$f(a) \to g(h(a), a) \to g(h(c), a)$ **is NOT basic**

- **Notation that presents inhibited positions**
  $f(a)^{\emptyset} \underset{\mathrm{b}}{\to} g(h(a), a)^{\{11,2\}} \underset{\mathrm{b}}{\to} g(b, a)^{\{2\}}$

Properties of basic reduction

- **Innermost sequence is basic, i.e.**

  $s_1 \underset{\mathsf{i}}{\to} s_2 \underset{\mathsf{i}}{\to} \cdots \underset{\mathsf{i}}{\to} s_n \; (\not\to\!\!\!\to s_{n+1})$ **implies**

  ${s_1}^{\emptyset} \underset{\mathsf{b}}{\to} {s_2}^{B_2} \underset{\mathsf{b}}{\to} \cdots \underset{\mathsf{b}}{\to} {s_n}^{B_n} \; (\overset{*}{\underset{\mathsf{b}}{\to}} {s_{n+1}}^{B_{n+1}})$

  for some $B_2, \ldots, B_n, (B_{n+1})$

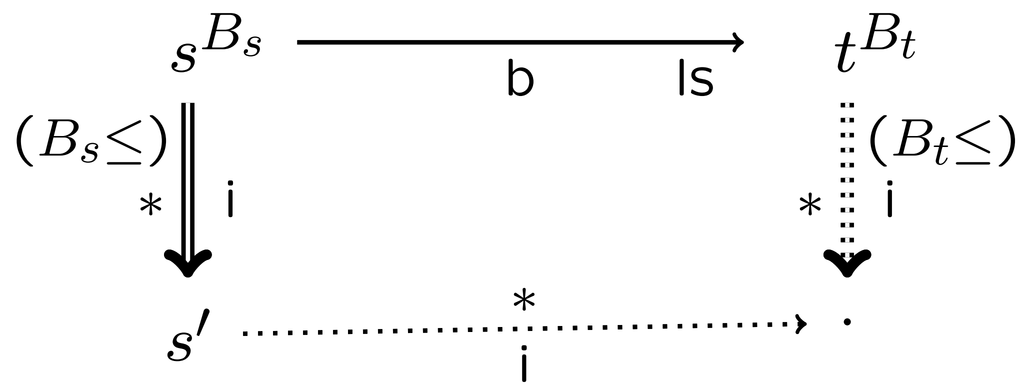
- **Basic sequence is stable under substitution;**

  ${s_1}^{B_1} \underset{\mathsf{b}}{\to} {s_2}^{B_2} \underset{\mathsf{b}}{\to} \cdots \underset{\mathsf{b}}{\to} {s_n}^{B_n}$ **implies**

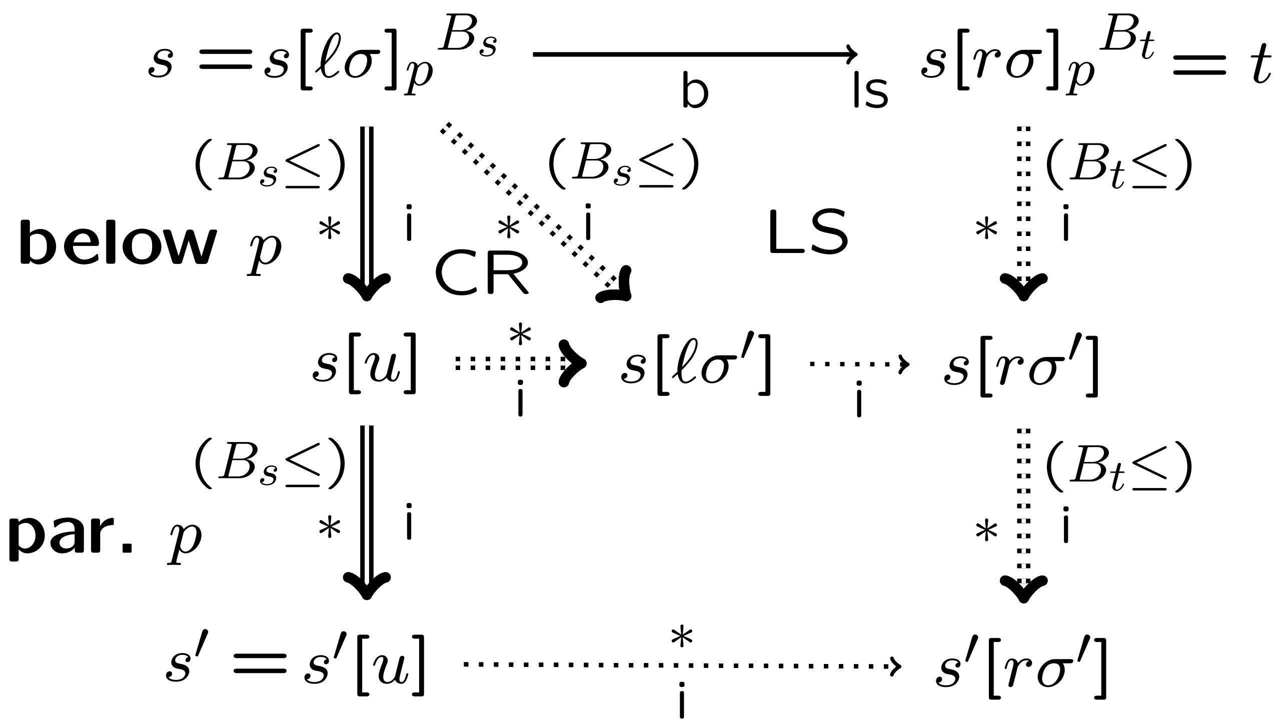
  $(s_1\theta)^{B_1} \underset{\mathsf{b}}{\to} (s_2\theta)^{B_2} \underset{\mathsf{b}}{\to} \cdots \underset{\mathsf{b}}{\to} (s_n\theta)^{B_n}$

Lemmas for proving the theorem

- **Lem.**

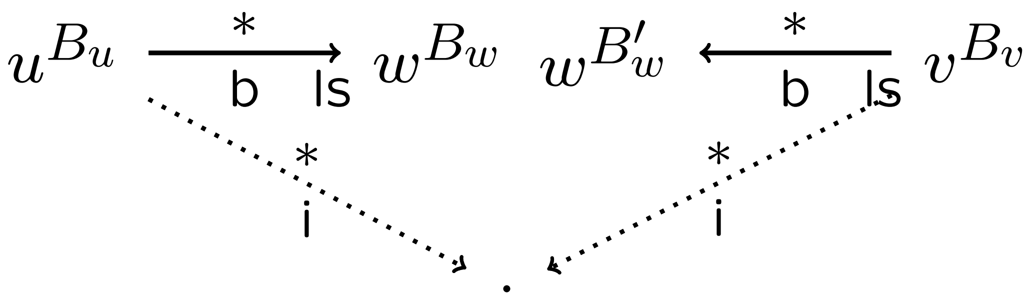


- **Proof:**

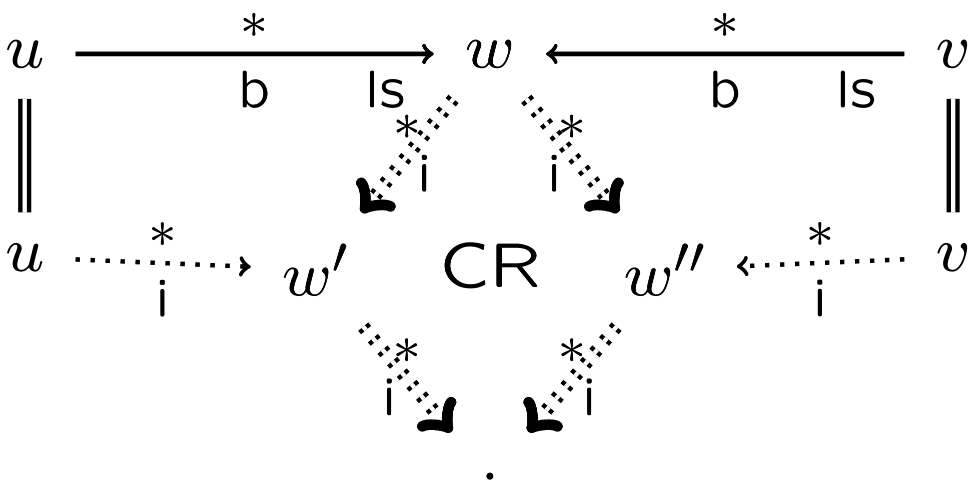


Lemmas for proving the theorem

- **Lem.**



- **Proof.**



Theorem and corollaries

- **Th. For** $\mathsf{SN}(\underset{\mathsf{i}}{\to})$**,**

  $$\mathsf{CP} \subseteq \xrightarrow[\mathsf{b}]{*}_{\mathsf{ls}} \cdot \xleftarrow[\mathsf{b}]{*}_{\mathsf{ls}} \implies \mathsf{CR}(\to)$$

- **Col. For** $\mathsf{SN}(\underset{\mathsf{i}}{\to})$**,**

  $$\mathsf{CP} \subseteq \xrightarrow[\mathsf{i}]{*}_{\mathsf{ls}} \cdot \xleftarrow[\mathsf{i}]{*}_{\mathsf{ls}} \implies \mathsf{CR}(\to)$$

- **Col. For** $\mathsf{SN}(\underset{\mathsf{i}}{\to})$**,**

  $$\mathsf{CP} \subseteq \xrightarrow[\mathsf{i}]{*}_{\mathsf{ls}} \cdot \overset{\|}{\longleftrightarrow}_{\mathsf{ls}} \cdot \xleftarrow[\mathsf{i}]{*}_{\mathsf{ls}} \implies \mathsf{CR}(\to)$$

Results obtained

- Open problem: Is confluence for innermost terminating TRSs decidable?
- A simple proof of undecidability of ground confluence for terminating TRSs
- Sufficient conditions of confluence for innermost terminating TRSs

$$\mathrm{CP} \subseteq \xrightarrow[\mathsf{gi}]{*} \cdot \leftrightarrow\!\!\!\!\!\!+\!\!+\;\; \cdot \xleftarrow[\mathsf{gi}]{*}, \qquad \mathrm{CP} \subseteq \xrightarrow[\mathsf{b}]{*}{}_{\mathsf{ls}} \cdot \xleftarrow[\mathsf{b}]{*}{}_{\mathsf{ls}}$$

Works left

- Solving the open problem above
- Better sufficient conditions (in case of the problem is undecidable)